JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 24, No. 6, November–December 2001

Efficient Dynamical Equations for Gyrostats

Paul C. Mitiguy*
MSC Software, San Mateo, California 94206
and
Keith J. Reckdahl†
Space Systems/Loral, Palo Alto, California 94303

To formulate equations of motion, the analyst must choose constants that characterize the mass distribution of system components. Traditionally, one chooses as constants the mass of each particle and the mass and central inertia scalars of each rigid body. However, this characterization of the mass distribution leads to inefficient equations of motion for gyrostats and necessitates the determination of an unnecessary large number of mass and inertia scalars. For gyrostats, there exist special formulas and a methodology for characterizing mass distribution that lead to efficient dynamic equations. In this context, efficient refers to relative simplicity, ease of manipulation for purposes of designing automatic control systems, and minimal consumption of computer time during numerical solution.

Nomenclature

a	=	distance from D to A_0, the mass center of A
${}^N\boldsymbol{a}^{G_0}$	=	acceleration of G_0 in N
$\boldsymbol{b}$	=	unit vector fixed in A and parallel to B's symmetry axis
b_1, b_2, b_3	=	parameters that locate B_0, the mass center of B from A_0
c_1, c_2, c_3	=	parameters that locate C_0, the mass center of C from A_0
${}^N\boldsymbol{F}^G$	=	effective force of G in N, $m^G\,{}^N\boldsymbol{a}^{G_0}$
g	=	distance from D to G_0, the mass center of G
I	=	moment of inertia of B about its center of mass
I^A	=	moment of inertia of A about a line passing through A_0 and parallel to $\boldsymbol{a}_3$
$I_{11}^A, I_{22}^A, I_{33}^A$	=	moments of inertia of A about A_0, the mass center of A, for $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$
$I_{12}^A, I_{13}^A, I_{23}^A$	=	products of inertia of A about A_0 for $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$
I^B, J^B	=	central moments of inertia of B for B's symmetry and transverse axes
I^C, J^C	=	central moments of inertia of C for C's symmetry and transverse axes
I^G	=	moment of inertia of G about a line passing through G_0 and parallel to $\boldsymbol{a}_3$
$\underline{\boldsymbol{I}}^{CG}$	=	dyadic associated with a cylindrical gyrostat, $\triangleq \underline{\boldsymbol{I}}^G - J\boldsymbol{bb}$
$\underline{\boldsymbol{I}}^G$	=	central inertia dyadic of G
$I_{11}^G, I_{22}^G, I_{33}^G$	=	moments of inertia of G about G_0, the mass center of G, for $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$
$I_{12}^G, I_{13}^G, I_{23}^G$	=	products of inertia of G about G_0 for $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$
$\underline{\boldsymbol{I}}^{SG}$	=	dyadic associated with a spherical gyrostat, $\triangleq \underline{\boldsymbol{I}}^G - I\,\underline{\boldsymbol{1}}$
J	=	moment of inertia of B about its symmetry axis
K	=	moment of inertia of B about its transverse axis
${}^N\boldsymbol{L}^G$	=	linear momentum of G in N, $m^G\,{}^N\boldsymbol{v}^{G_0}$
m^A, m^B, m^C	=	mass of A, B, and C
m^G	=	mass of G
${}^N\boldsymbol{v}^{G_0}$	=	velocity of G_0, the mass center of G, in N
${}^N\boldsymbol{v}_r^{G_0}$	=	rth partial velocity of G_0 in N
${}^N\boldsymbol{\alpha}^A$	=	angular acceleration of rigid body B in reference frame N
${}^A\boldsymbol{\omega}_r^B$	=	rth partial angular velocity of rigid body B in reference frame A
${}^N\boldsymbol{\omega}^A$	=	angular velocity of rigid body A in reference frame N
$\underline{\boldsymbol{1}}$	=	unit dyadic

I. Introduction

AS early as 1744, when the mechanic Serson built a device for determining an artificial horizon at sea, systems containing gyrostats have been in widespread use. For example, in 1883, Lord Kelvin, who coined the word gyrostat, made practical and theoretical contributions to gyrodevices with his study of the gyrocompass. In 1904, the utility of gyros as motion stabilizing devices was successfully demonstrated by Otto Schlick, who used a gyro to limit the rolling of a ship at sea. Over the last century, gyrostats have been used extensively as orientation and rate sensors in watercraft, aircraft, satellites, and targeting devices and as stabilizers onboard ships and satellites. Because of their widespread use, the ability to formulate efficient equations of motion for systems containing gyrostats has become increasingly important, especially in situations requiring real-time numerical solutions.

Two classes of gyrostats frequently encountered in engineering practice are cylindrical gyrostats and spherical gyrostats. Each such gyrostat G consists of a rigid body A connected to a rigid body B in such a way that B_0, the mass center of B, is fixed in A. The term cylindrical gyrostat is applicable when B is inertially axisymmetric and has a simple angular velocity in A in a direction parallel to the symmetry axis (Fig. 1 shows a representative cylindrical gyrostat). (Note that a body B is inertially axisymmetric if it has the same moment of inertia about all lines that pass through B_0 and are perpendicular to the symmetry axis.) Examples of mechanical systems that contain cylindrical gyrostats include systems having components such as rotors, wheels, screws, or propellers with three or more evenly spaced blades. The term spherical gyrostat applies when B has the same moment of inertia about all lines passing through B_0 and when B has three rotational degrees of freedom in A. (Fig. 2 shows a representative spherical gyrostat). One example of a mechanical system that contains a spherical gyrostat is a satellite with a spherical damper.

In this paper, it is shown how formulas for five important dynamic quantities, namely, angular momentum, effective/inertia torque, kinetic energy, generalized momentum, and generalized effective/inertia force, can be efficiently formed for systems containing cylindrical or spherical gyrostats. To gain an appreciation for the differences between a traditional and efficient approach for analyzing gyrostats, a variety of dynamic expressions, operations counts, and timing results are presented for a simplified helicopter example.

Received 2 August 2000; revision received 27 December 2000; accepted for publication 27 December 2000.

*Principal Technical Developer, Research and Development, 66 Bovet Road.

†Engineering Specialist, Dynamics and Controls Analysis, 3825 Fabian Way.

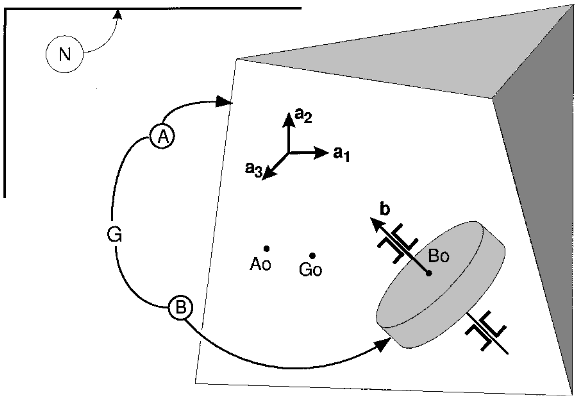

Fig. 1 Cylindrical gyrostat.

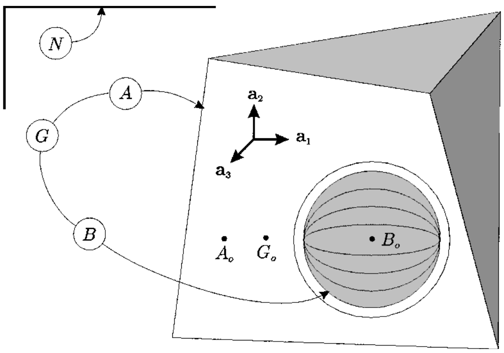

Fig. 2 Spherical gyrostat.

Subsequently, the formulas that enable the efficient approach are provided for the calculation of each of the five dynamic quantities. Next, the utility of these equations is demonstrated in the context of two more examples. Finally, a derivation of these equations is presented.

Several notational practices are used throughout this paper. Boldface symbols denote vectors and bold-faced underlined symbols denote dyadics. The angular velocity of a rigid body, for example, B, in a reference frame, for example, A, is denoted ${}^A\boldsymbol{\omega}^B$. Similarly, the velocity of a point, for example, P, in a reference frame, for example, A is denoted ${}^A\boldsymbol{v}^P$. Partial velocities have a right subscript that is associated with a motion variable (generalized speed). (The term partial velocity may be unfamiliar to readers who have not studied Kane's method. The partial velocity of body B in reference frame N associated with the motion variable ω_r can be calculated by ${}^A\boldsymbol{\omega}^B_r = \partial{}^N\boldsymbol{\omega}^B/\partial\omega_r$).

II. Motivating Example: Two-Rotor Helicopter

The formulation of equations of motion of a system S in a Newtonian reference frame N necessitates the selection of parameters that characterize the mass distribution of S. At times, S or a portion of S comprises a gyrostat G, defined as a collection of rigid bodies whose mass distribution properties (mass center location and central inertia scalars) are time invariant in one of the rigid bodies of G, called the carrier. For example, Fig. 3 is a schematic representation of a simplified helicopter consisting of a fuselage A, a main rotor B, and a tail rotor C. Together, A, B, and C form a gyrostat G, with A the carrier. (Helicopters such as the Bell 206 helicopter do not qualify as a gyrostat because their main rotor and tail rotor both have two blades. To qualify as a gyrostat, the rotors must have three or more evenly spaced blades, for example, a Sikorsky HH-35D.)

The mass distribution of the helicopter G in Fig. 3 may be characterized in several ways. One way to characterize the mass distribution of G is with the 19 constants: m^A, m^B, m^C, I^A_{11}, I^A_{22}, I^A_{33}, I^A_{12}, I^A_{13}, I^A_{23}, I^B, J^B, I^C, J^C, b_1, b_2, b_3, c_1, c_2, and c_3.

This particular characterization of mass distribution is the one used by popular numerical commercial multibody programs, for example, Adams (Mechanical Dynamics, Inc.), MSC.visualNastran (MSC. Software), Pro/Mechanica (Parametric Technology Corporation), Dads (Cadsi), as well as the characterization used by symbolic commercial multibody programs described in the literature, for example, AutoSim[1] (Mechanical Solutions Corporation) and Autolev[2] (OnLine Dynamics, Inc.) for which reason we call it traditional. (Several of the formulas presented in this paper were recently adopted in Autolev 3.0.)

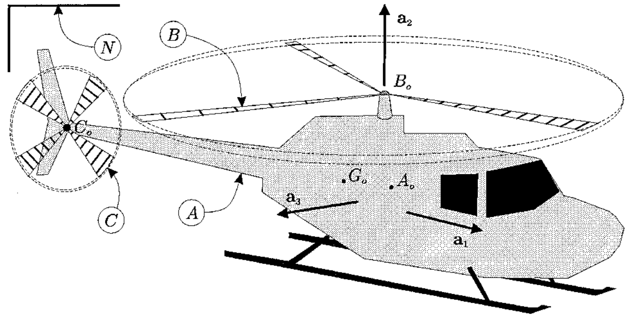

Fig. 3 Helicopter with two axisymmetric rotors.

Alternatively, as will be seen, it is possible to formulate equations of motion with significantly fewer mass distribution constants. Specifically, one may characterize the mass distribution of G with 9 constants instead of 19, namely, m^G, I^G_{11}, I^G_{22}, I^G_{33}, I^G_{12}, I^G_{13}, I^G_{23}, I^B, and I^C.

To show that the characterization of the helicopter's mass distribution can have a profound effect on the complexity of the equations governing the helicopter's motion, some of the dynamic differential equations are given in Appendices A and B. These equations relate the mass distribution constants; the time derivatives of the motion variables ω_1, ω_2, ω_3, ω^B, ω^C, v_1, v_2, and v_3; and the parameters F_r $(r = 1, \ldots, 8)$ that characterize aerodynamic, gravitational, and motor forces. (The quantities F_r $(r = 1, \ldots, 8)$ are called generalized active forces and depend on time and the state.) To make Appendices A and B as compact as possible, the motion variables were chosen to deal with the rotational motions of G in N in the most efficient way known.[3] Specifically,

$$\omega_i \triangleq {}^N\boldsymbol{\omega}^A \cdot \boldsymbol{a}_i \qquad (i = 1, 2, 3)$$

$$\omega^B \triangleq {}^N\boldsymbol{\omega}^B \cdot \boldsymbol{a}_2, \qquad \omega^C \triangleq {}^N\boldsymbol{\omega}^C \cdot \boldsymbol{a}_3$$

where ${}^N\boldsymbol{\omega}^A$, ${}^N\boldsymbol{\omega}^B$, and ${}^N\boldsymbol{\omega}^C$, are the angular velocities of A in N, B in N, and C in N, respectively. The motion variables v_1, v_2, and v_3, which characterize translational motions of the helicopter, may be defined in a variety of ways. One definition that seems natural when the traditional mass distribution constants are used is

$$v_i \triangleq {}^N\boldsymbol{v}^{A_0} \cdot \boldsymbol{a}_i \qquad (i = 1, 2, 3)$$

where ${}^N\boldsymbol{v}^{A_0}$ is the velocity of A_0 in N. Conversely, when the new mass distribution constants are used, the following definitions are natural:

$$v_i \triangleq {}^N\boldsymbol{v}^{G_0} \cdot \boldsymbol{a}_i \qquad (i = 1, 2, 3)$$

where ${}^N\boldsymbol{v}^{G_0}$ is the velocity of G_0, the mass center of G, in N.

Appendix A shows less than one-third of one of the explicit dynamic differential equations when the traditional mass distribution constants are used. Obviously, the equation governing $\dot{\omega}_1$ is very long. Moreover, the traditional equations for $\dot{\omega}_2$ and $\dot{\omega}_3$ are equally long, and those governing $\dot{v}_1$, $\dot{v}_2$, and $\dot{v}_3$ are even longer. The traditional equations for $\dot{\omega}^B$ and $\dot{\omega}^C$ are each less than one line long, their brevity being directly attributable to the use of the motion variables advocated in Ref. 3. Appendix B shows all eight of the dynamic differential equations when the new mass distribution constants are used. As can be seen at a glance, the first dynamic differential equation in Appendix B is much shorter than its counterpart in Appendix A. The differences in length between the new and traditional dynamic differential equations for $\dot{\omega}_2$, $\dot{\omega}_3$, $\dot{v}_1$, $\dot{v}_2$, and $\dot{v}_3$ are equally striking.

Table 1 Helicopter operations count when employing intermediate variables

Operation	Traditional	New
Addition	27	8
Subtraction	85	32
Multiplication	145	59
Division	8	8
Assignment	60	28
Total	325	135

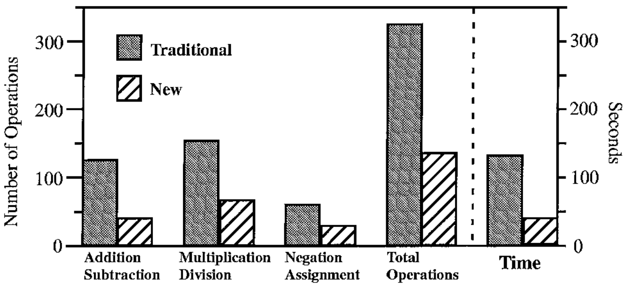

Fig. 4 Helicopter efficiency comparison chart when employing intermediate variables.

Of course, in producing motion simulations, no one is likely to write dynamic equations in their fully expanded form Appendices A and B because this engenders repetitious calculations. A more efficient approach is to introduce intermediate variables for the specific purpose of reducing the number of calculations at each integration step. Such variables are introduced automatically when the symbol manipulation program Autolev[2] is employed to formulate the equations of motion. Once the equations are in hand, one can count, for each of the two sets of equations of motion, the number of operations required to evaluate the time derivatives of the motion variables ($\dot\omega_1$, $\dot\omega_2$, $\dot\omega_3$, $\dot\omega^B$, $\dot\omega^C$, $\dot v_1$, $\dot v_2$, and $\dot v_3$) at each time step. For each set of equations, such counts were performed, and the results are given in Table 1 under the headings Traditional and New. The results clearly show the computational superiority of the new choice of constants to the traditional one.

A related, corroborative finding is that 150,000 evaluations of the time derivatives of the motion variables required 131.5 s when the traditional constants were used, but only 38.0 s when the new constants were employed (calculations performed on an 386-25-MHz computer). Concomitantly, use of the new approach substantially reduces the computational time required for numerical motion simulation. A clear picture of the superiority of the new choice of constants to the traditional ones is provided in Fig. 4, which compares the operations and timing results in the form of a chart.

III. Cylindrical Gyrostats

Figure 1 is a schematic representation of a gyrostat G that consists of a carrier A and an axisymmetric rotor B. For purposes of analyzing motions of G in a reference frame N, it may be necessary to form expressions for G's central angular momentum, effective/inertia torque, kinetic energy, generalized momentum, and generalized effective/inertia force.

Several mass distribution and motion quantities that are useful in the formulas that follow are listed in the Nomenclature.

In the next section, formulas for five dynamic quantities are presented for cylindrical gyrostats. Two sets of formulas are presented for each dynamic quantity. Although the formulas are analytically equivalent, the first set of formulas is advantageous when one uses conventional motion variables to describe the rotor's angular motions. The second set of formulas is superior when one employs the motion variables in Ref. 3 to describe the rotor's angular motions. In general, the first set of formulas for each dynamic quantity is more efficient when one is analyzing fixed rotors, meaning rotors that have a prescribed (known) motion relative to their carriers. The second set of formulas for each dynamic quantity is more efficient when analyzing free rotors, meaning rotors whose motions are governed by linear or differential dynamic equations. The utility of treating fixed and free rotors differently was first noticed at the end of an analysis in Ref. 4.

Accompanying each set of formulas is an equation that implements a specific definition for the variable associated with the rotor's motion. To make the present work self-contained, the two choices for the rotor's motion variable are defined as follows.

Conventional motion variable:

$$ {}^A\omega^B \triangleq {}^A\boldsymbol{\omega}^B \cdot \boldsymbol{b} \tag{1} $$

Motion variable in Ref. 3:

$$ {}^N\omega^B \triangleq {}^N\boldsymbol{\omega}^B \cdot \boldsymbol{b} \tag{2} $$

A. Formulas for Cylindrical Gyrostats

1. Angular Momentum

An efficient way to form ${}^N\boldsymbol{H}^{G/G_0}$, the central angular momentum of a cylindrical gyrostat G in a reference frame N, is

$$ {}^N\boldsymbol{H}^{RG/G_0} \triangleq \underline{\boldsymbol{I}}^G \cdot {}^N\boldsymbol{\omega}^A \tag{3} $$

$$ \begin{aligned} {}^N\boldsymbol{H}^{G/G_0} &= {}^N\boldsymbol{H}^{RG/G_0} + J\,{}^A\boldsymbol{\omega}^B \\ &\underset{(1)}{=} {}^N\boldsymbol{H}^{RG/G_0} + J\,{}^A\omega^B\boldsymbol{b} \end{aligned} \tag{4} $$

(Equation numbers appearing under an equal sign or under terms refer to equations numbered correspondingly.) ${}^N\boldsymbol{H}^{G/G_0}$ can also be calculated with a different set of formulas. This second set of formulas is superior for free rotors if one uses the motion variable in Eq. (2):

$$ {}^N\boldsymbol{H}^{CG/G_0} \triangleq \underline{\boldsymbol{I}}^{CG} \cdot {}^N\boldsymbol{\omega}^A \tag{5} $$

$$ \begin{aligned} {}^N\boldsymbol{H}^{G/G_0} &= {}^N\boldsymbol{H}^{CG/G_0} + J\boldsymbol{bb} \cdot {}^N\boldsymbol{\omega}^B \\ &\underset{(2)}{=} {}^N\boldsymbol{H}^{CG/G_0} + J\,{}^N\omega^B\boldsymbol{b} \end{aligned} \tag{6} $$

2. Effective/Inertia Torque

An efficient way to form ${}^N\boldsymbol{T}^G$, the effective torque of a cylindrical gyrostat G in a reference frame N, is

$$ {}^N\boldsymbol{T}^{RG} \triangleq \underline{\boldsymbol{I}}^G \cdot {}^N\boldsymbol{\alpha}^A + {}^N\boldsymbol{\omega}^A \times \underline{\boldsymbol{I}}^G \cdot {}^N\boldsymbol{\omega}^A \tag{7} $$

$$ \begin{aligned} {}^N\boldsymbol{T}^G &= {}^N\boldsymbol{T}^{RG} + J({}^A\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B) \\ &\underset{(1)}{=} {}^N\boldsymbol{T}^{RG} + J({}^A\dot\omega^B\boldsymbol{b} + {}^N\boldsymbol{\omega}^A \times {}^A\omega^B\boldsymbol{b}) \end{aligned} \tag{8} $$

A second set of formulas for ${}^N\boldsymbol{T}^G$, which is superior for free rotors when one uses the motion variable in Eq. (2), is

$$ {}^N\boldsymbol{T}^{CG} \triangleq \underline{\boldsymbol{I}}^{CG} \cdot {}^N\boldsymbol{\alpha}^A + {}^N\boldsymbol{\omega}^A \times \underline{\boldsymbol{I}}^{CG} \cdot {}^N\boldsymbol{\omega}^A \tag{9} $$

$$ \begin{aligned} {}^N\boldsymbol{T}^G &= {}^N\boldsymbol{T}^{CG} + J(\boldsymbol{bb} \cdot {}^N\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times \boldsymbol{bb} \cdot {}^N\boldsymbol{\omega}^B) \\ &\underset{(2)}{=} {}^N\boldsymbol{T}^{CG} + J({}^N\dot\omega^B\boldsymbol{b} + {}^N\boldsymbol{\omega}^A \times {}^N\omega^B\boldsymbol{b}) \end{aligned} \tag{10} $$

The inertia torque of G in an implied reference frame N is denoted $\boldsymbol{T}^*$ and is defined

$$ \boldsymbol{T}^* \triangleq -{}^N\boldsymbol{T}^G \tag{11} $$

3. Kinetic Energy

An efficient way to form ${}^N\boldsymbol{K}^G$, the kinetic energy of a cylindrical gyrostat G in a reference frame N, is

$$ {}^NK^{RG} \triangleq \tfrac{1}{2}m^G\,{}^N\boldsymbol{v}^{G_0} \cdot {}^N\boldsymbol{v}^{G_0} + \tfrac{1}{2}{}^N\boldsymbol{\omega}^A \cdot \underline{\boldsymbol{I}}^G \cdot {}^N\boldsymbol{\omega}^A \tag{12} $$

$$ \begin{aligned} {}^NK^G &= {}^NK^{RG} + \tfrac{1}{2}J\,{}^A\boldsymbol{\omega}^B \cdot {}^A\boldsymbol{\omega}^B + J\,{}^N\boldsymbol{\omega}^A \cdot {}^A\boldsymbol{\omega}^B \\ &\underset{(1)}{=} {}^NK^{RG} + \tfrac{1}{2}J\,{}^A\omega^{B^2} + J\,{}^N\boldsymbol{\omega}^A \cdot {}^A\boldsymbol{\omega}^B\boldsymbol{b} \end{aligned} \tag{13} $$

A second set of formulas for ${}^N K^G$, which is superior for free rotors when one uses the motion variable in Eq. (2), is

$$ {}^N K^{CG} \triangleq \tfrac{1}{2} m^G\, {}^N\boldsymbol{v}^{G_0} \cdot {}^N\boldsymbol{v}^{G_0} + \tfrac{1}{2}\, {}^N\boldsymbol{\omega}^A \cdot \underline{\boldsymbol{I}}^{CG} \cdot {}^N\boldsymbol{\omega}^A \qquad (14) $$

$$ {}^N K^G = {}^N K^{CG} + \tfrac{1}{2} J \left({}^N\boldsymbol{\omega}^B \cdot \boldsymbol{b}\right)^2 \underset{(2)}{=} {}^N K^{CG} + \tfrac{1}{2} J\, {}^N\omega^{B^2} \qquad (15) $$

4. Generalized Momentum

An efficient way to form ${}^N L_r^G$, the rth generalized momentum of a cylindrical gyrostat G in a reference frame N, is

$$ {}^N L_r^{RG} \triangleq {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{L}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{H}^{RG/G_0} \qquad (16) $$

$$ {}^N L_r^G = {}^N L_r^{RG} + J\left({}^N\boldsymbol{\omega}_r^A \cdot {}^A\boldsymbol{\omega}^B + {}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\omega}^B\right) \qquad (17) $$

(Note that the subscript r appearing throughout the remainder of this paper denotes an association with the rth motion variable.) Equations (16) and (17) are more efficient for fixed rotors than free rotors because, when ${}^A\boldsymbol{\omega}^B$ is prescribed (specified), ${}^A\boldsymbol{\omega}_r^B = \boldsymbol{0}$ and the last term in parentheses in Eq. (17) is 0. A second set of formulas for ${}^N L_r^G$, which is superior for free rotors, is

$$ {}^N L_r^{CG} \triangleq {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{L}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{H}^{CG/G_0} \qquad (18) $$

$$ {}^N L_r^G = {}^N L_r^{CG} + J\left({}^N\boldsymbol{\omega}_r^B \cdot \boldsymbol{b}\right)\left({}^N\boldsymbol{\omega}^B \cdot \boldsymbol{b}\right) \underset{(2)}{=} {}^N L_r^{CG} + J\left({}^N\boldsymbol{\omega}_r^B \cdot \boldsymbol{b}\right) {}^N\omega^B \qquad (19) $$

Note that when one uses the motion variable in Eq. (2), the last term in Eq. (19) depends on r and simplifies to either $J\,{}^N\omega^B$ or 0.

5. Generalized Effective/Inertia Force

An efficient way to form ${}^N F_r^G$, the rth generalized effective force of a cylindrical gyrostat G in a reference frame N, is

$$ {}^N F_r^{RG} \triangleq {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{T}^{RG} \qquad (20) $$

$$ {}^N F_r^G = {}^N F_r^{RG} + J\Big[{}^N\boldsymbol{\omega}_r^A \cdot \left({}^A\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B\right) + {}^A\boldsymbol{\omega}_r^B \cdot \left({}^N\boldsymbol{\alpha}^A + {}^A\boldsymbol{\alpha}^B\right)\Big] \qquad (21) $$

This first set of formulas is more efficient for fixed rotors than free rotors because, when ${}^A\boldsymbol{\omega}^B$ is prescribed (specified), ${}^A\boldsymbol{\omega}_r^B = \boldsymbol{0}$ and the last term in parentheses in Eq. (21) is 0. A second set of formulas for ${}^N F_r^G$, which is superior for free rotors, is

$$ {}^N F_r^{CG} \triangleq {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{T}^{CG} \qquad (22) $$

$$ {}^N F_r^G = {}^N F_r^{CG} + J\Big[{}^N\boldsymbol{\omega}_r^A \cdot \left({}^N\boldsymbol{\omega}^A \times \boldsymbol{bb} \cdot {}^N\boldsymbol{\omega}^B\right) + {}^N\boldsymbol{\omega}_r^B \cdot \boldsymbol{bb} \cdot {}^N\boldsymbol{\alpha}^B\Big] \underset{(2)}{=} {}^N F_r^{CG} + J\Big[{}^N\boldsymbol{\omega}_r^A \cdot \left({}^N\boldsymbol{\omega}^A \times {}^N\omega^B \boldsymbol{b}\right) + {}^N\boldsymbol{\omega}_r^B \cdot {}^N\dot{\omega}^B \boldsymbol{b}\Big] \qquad (23) $$

Note that when one uses the motion variable in Eq. (2), the last term in parentheses in Eq. (23) depends on r and simplifies to either ${}^N\dot{\omega}^B$ or 0.

The generalized inertia force of G in an implied reference frame N is denoted F_r^* and is defined

$$ F_r^* \triangleq -{}^N F_r^G \qquad (24) $$

B. Cylindrical Gyrostat Example: Two Wheeled Vehicle

Figure 5 is a schematic representation of a vehicle G, formed by a rigid chassis A, that carries two identical axisymmetric circular wheels, B and C, each of radius R. This example considers a vehicle with two rolling wheels rather than four because the other two wheels (missing in Fig. 5) are considered to be skidding and their mass and inertia properties are included in A. Equations of motion are to be formulated on the basis of the assumptions that B and C roll without slipping on a horizontal plane N and are completely free to rotate relative to A on an axle whose midpoint D is located a distance b from each wheel. To describe the motion of G in N, the variables ω and v are defined as

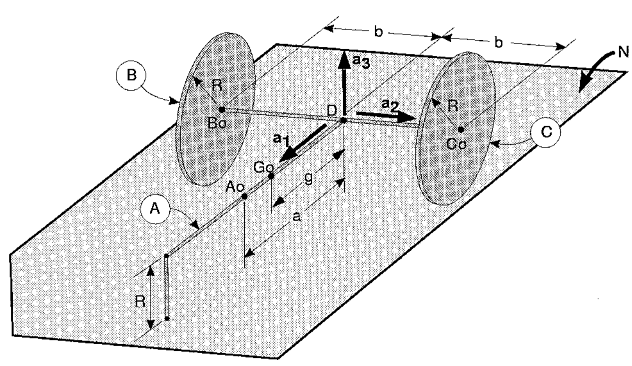

Fig. 5 Vehicle schematic.

$$ \omega \triangleq {}^N\boldsymbol{\omega}^A \cdot \boldsymbol{a}_3, \qquad v \triangleq {}^N\boldsymbol{v}^D \cdot \boldsymbol{a}_1 $$

where ${}^N\boldsymbol{\omega}^A$ is the angular velocity of A in N, ${}^N\boldsymbol{v}^D$ is the velocity of D in N, and $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$ are unit vectors fixed in A with $\boldsymbol{a}_3$ perpendicular to N, $\boldsymbol{a}_2$ parallel to the axle of A, and $\boldsymbol{a}_1 = \boldsymbol{a}_2 \times \boldsymbol{a}_3$.

To characterize the mass distribution of G in the new manner, one may introduce four constants, namely, m^G, g, I^G, and J. With the aid of Eq. (21), ${}^N F_\omega^G$ and ${}^N F_v^G$, the generalized effective forces associated with the motion variables ω and v, are

$$ {}^N F_\omega^G = \left(2Jb^2/R^2 + I^G + m^G g^2\right)\dot{\omega} + m^G g \omega v $$

$$ {}^N F_v^G = \left(2J/R^2 + m^G\right)\dot{v} - m^G g \omega^2 $$

so that, when the generalized active forces on the vehicle are zero, the new dynamic differential equations may be written as

$$ \dot{\omega} = \frac{-m^G g}{2Jb^2/R^2 + I^G + m^G g^2}\,\omega v, \qquad \dot{v} = \frac{m^G g}{2J/R^2 + m^G}\,\omega^2 $$

The following six constants characterize the mass distribution of G in a more traditional manner: m^A, m^B, a, I^A, J, and K. With the aid of the formula for the generalized effective force of a rigid body [see Ref. 5, page 125, Eq. (4.11.7)], applied separately to A, B, and C, the traditional dynamic differential equations are found to be

$$ \dot{\omega} = \frac{-m^A a}{2Jb^2/R^2 + I^A + 2K + m^A a^2 + 2m^B b^2}\,\omega v $$

$$ \dot{v} = \frac{m^A a}{2J/R^2 + m^A + 2m^B}\,\omega^2 $$

The equivalence of the new and traditional dynamic differential equations becomes apparent after observing that

$$ m^G g = m^A a, \qquad m^G = m^A + 2m^B $$

$$ I^G + m^G g^2 = I^A + 2K + m^A a^2 + 2m^B b^2 $$

Even though these two sets of equations are equivalent, the new equations enjoy two substantial advantages over their traditional counterparts. First, with the new formula for generalized effective force [Eq. (21)], fewer vector operations are required to form expressions for $\dot{\omega}$ and $\dot{v}$. Second, the new equations are simpler and require fewer calculations to generate numerical values for $\dot{\omega}$ and $\dot{v}$. The results of this example are much less dramatic than those of the helicopter example and were purposely chosen to show clearly the equivalence of the traditional and new methods.

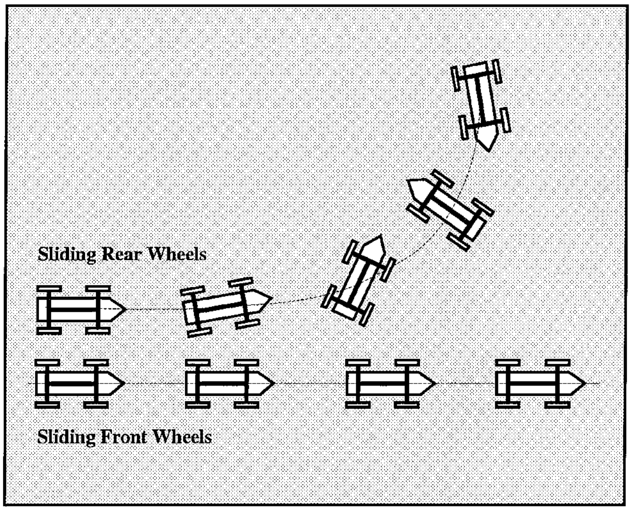

Fig. 6 Vehicle skid.

Although this example has served its purpose in demonstrating the equivalence and difference between the traditional and efficient methods for generating equations of motion for systems containing cylindrical gyrostats, this system has interesting motions. Numerical integration of the efficient dynamic differential equations were performed with the following vehicle parameters, mass distribution constants, and initial values:

$$b = 0.75 \text{ m} \qquad R = 0.35 \text{ m} \qquad g = 1.5 \text{ m}$$

$$m^G = 700 \text{ kg} \qquad J = 2.0 \text{ kg}\cdot\text{m}^2 \qquad I^G = 350 \text{ kg}\cdot\text{m}^2$$

$$\omega(0) = 0.01 \text{ rad/s} \qquad v(0) = \pm 0.25 \text{ m/s}$$

Figure 6 is a graphical representation of 1 s of motion for two simulations. The first (top) simulation corresponds to $v(0) = -0.25$ m/s (sliding rear wheels, rolling front wheels) and the second (bottom) simulation corresponds to $v(0) = +0.25$ m/s (sliding front wheels, rolling rear wheels). As is apparent, the straight-line motion of a vehicle with sliding rear wheels is unstable, whereas with sliding front wheels, it is stable.

IV. Spherical Gyrostats

Figure 2 is a schematic representation of a gyrostat G that consists of a carrier A and a spherical rotor B. For purposes of analyzing motions of G in a reference frame N, it may be necessary to form expressions for G's central angular momentum, effective/inertia torque, kinetic energy, generalized momentum, and generalized effective/inertia force.

Several mass distribution and motion quantities that are useful in the formulas that follow are m^G, $\underline{\boldsymbol{I}}^G$, I, $\underline{\boldsymbol{1}}$, $\underline{\boldsymbol{I}}^{SG}$, ${}^N\boldsymbol{L}^G$, and ${}^N\boldsymbol{F}^G$.

In the next section, two sets of formulas are presented for spherical gyrostats. Although the two sets of formulas are analytically equivalent, the first set is advantageous when analyzing fixed rotors and using conventional motion variables to describe the rotor's angular motions. The second set is superior when analyzing free rotors and using the motion variables in Ref. 3 to describe the rotor's angular motions. To make the present work self-contained, the two choices for the rotor's motion variables are defined and their corresponding angular velocity/acceleration expressions are given.

Conventional motion variables:

$$
\begin{aligned}
{}^A\omega_i^B &\triangleq {}^A\boldsymbol{\omega}^B \cdot \boldsymbol{a}_i \qquad (i = 1, 2, 3)\\
{}^A\boldsymbol{\omega}^B &= {}^A\omega_1^B\boldsymbol{a}_1 + {}^A\omega_2^B\boldsymbol{a}_2 + {}^A\omega_3^B\boldsymbol{a}_3\\
{}^A\boldsymbol{\alpha}^B &= {}^A\dot{\omega}_1^B\boldsymbol{a}_1 + {}^A\dot{\omega}_2^B\boldsymbol{a}_2 + {}^A\dot{\omega}_3^B\boldsymbol{a}_3
\end{aligned}
\tag{25}
$$

Motion variables in Ref. 3:

$$
\begin{aligned}
{}^N\omega_i^B &\triangleq {}^N\boldsymbol{\omega}^B \cdot \boldsymbol{a}_i \qquad (i = 1, 2, 3)\\
{}^N\boldsymbol{\omega}^B &= {}^N\omega_1^B\boldsymbol{a}_1 + {}^N\omega_2^B\boldsymbol{a}_2 + {}^N\omega_3^B\boldsymbol{a}_3\\
{}^N\boldsymbol{\alpha}^B &= {}^N\dot{\omega}_1^B\boldsymbol{a}_1 + {}^N\dot{\omega}_2^B\boldsymbol{a}_2 + {}^N\dot{\omega}_3^B\boldsymbol{a}_3
\end{aligned}
\tag{26}
$$

A. Formulas for Spherical Gyrostats

1. *Angular Momentum*

An efficient way to form ${}^N\boldsymbol{H}^{G/G_0}$, the central angular momentum of a spherical gyrostat G in a reference frame N, is

$$ {}^N\boldsymbol{H}^{RG/G_0} \triangleq \underline{\boldsymbol{I}}^G \cdot {}^N\boldsymbol{\omega}^A \tag{27}$$

$$ {}^N\boldsymbol{H}^{G/G_0} = {}^N\boldsymbol{H}^{RG/G_0} + I\,{}^A\boldsymbol{\omega}^B \tag{28}$$

${}^N\boldsymbol{H}^{G/G_0}$ can also be calculated with a different set of formulas. This second set of formulas is superior for free rotors if one uses the motion variable in Eq. (26):

$$ {}^N\boldsymbol{H}^{SG/G_0} \triangleq \underline{\boldsymbol{I}}^{SG} \cdot {}^N\boldsymbol{\omega}^A \tag{29}$$

$$ {}^N\boldsymbol{H}^{G/G_0} = {}^N\boldsymbol{H}^{SG/G_0} + I\,{}^N\boldsymbol{\omega}^B \tag{30}$$

2. *Effective/Inertia Torque*

An efficient way to form ${}^N\boldsymbol{T}^G$, the effective torque of a spherical gyrostat G in a reference frame N, is

$$ {}^N\boldsymbol{T}^{RG} \triangleq \underline{\boldsymbol{I}}^G \cdot {}^N\boldsymbol{\alpha}^A + {}^N\boldsymbol{\omega}^A \times \underline{\boldsymbol{I}}^G \cdot {}^N\boldsymbol{\omega}^A \tag{31}$$

$$ {}^N\boldsymbol{T}^G = {}^N\boldsymbol{T}^{RG} + I\,({}^A\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B) \tag{32}$$

A second set of formulas for ${}^N\boldsymbol{T}^G$, which is superior for free rotors when one uses the motion variables in Eq. (26), is

$$ {}^N\boldsymbol{T}^{SG} \triangleq \underline{\boldsymbol{I}}^{SG} \cdot {}^N\boldsymbol{\alpha}^A + {}^N\boldsymbol{\omega}^A \times \underline{\boldsymbol{I}}^{SG} \cdot {}^N\boldsymbol{\omega}^A \tag{33}$$

$$ {}^N\boldsymbol{T}^G = {}^N\boldsymbol{T}^{SG} + I\,{}^N\boldsymbol{\alpha}^B \tag{34}$$

The inertia torque of G in an implied reference frame N is denoted $\boldsymbol{T}^*$ and is defined as

$$ \boldsymbol{T}^* \triangleq -{}^N\boldsymbol{T}^G \tag{35}$$

3. *Kinetic Energy*

An efficient way to form ${}^N K^G$, the kinetic energy of a spherical gyrostat G in a reference frame N, is

$$ {}^N K^{RG} \triangleq \tfrac{1}{2} m^G\,{}^N\boldsymbol{v}^{G_0} \cdot {}^N\boldsymbol{v}^{G_0} + \tfrac{1}{2}\,{}^N\boldsymbol{\omega}^A \cdot \underline{\boldsymbol{I}}^G \cdot {}^N\boldsymbol{\omega}^A \tag{36}$$

$$ {}^N K^G = {}^N K^{RG} + \tfrac{1}{2} I\,{}^A\boldsymbol{\omega}^B \cdot {}^A\boldsymbol{\omega}^B + I\,{}^N\boldsymbol{\omega}^A \cdot {}^A\boldsymbol{\omega}^B \tag{37}$$

A second set of formulas for ${}^N K^G$, which is superior for free rotors when one uses the motion variables in equation (26), is

$$ {}^N K^{SG} \triangleq \tfrac{1}{2} m^G\,{}^N\boldsymbol{v}^{G_0} \cdot {}^N\boldsymbol{v}^{G_0} + \tfrac{1}{2}\,{}^N\boldsymbol{\omega}^A \cdot \underline{\boldsymbol{I}}^{SG} \cdot {}^N\boldsymbol{\omega}^A \tag{38}$$

$$ {}^N K^G = {}^N K^{SG} + \tfrac{1}{2} I\,{}^N\boldsymbol{\omega}^B \cdot {}^N\boldsymbol{\omega}^B \tag{39}$$

4. *Generalized Momentum*

An efficient way to form ${}^N L_r^G$, the rth generalized momentum of a spherical gyrostat G in a reference frame N, is

$$ {}^N L_r^{RG} \triangleq {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{L}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{H}^{RG/G_0} \tag{40}$$

$$ {}^N L_r^G = {}^N L_r^{RG} + I\left({}^N\boldsymbol{\omega}_r^A \cdot {}^A\boldsymbol{\omega}^B + {}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\omega}^B\right) \tag{41}$$

Equation (40) and (41) are more efficient for fixed rotors than free rotors because, when ${}^A\boldsymbol{\omega}^B$ is prescribed (specified), ${}^A\boldsymbol{\omega}_r^B = \boldsymbol{0}$ and the last term in parentheses in Eq. (41) is 0. A second set of formulas for ${}^N L_r^G$, which is superior for free rotors, is

$$ {}^N L_r^{SG} \triangleq {}^N \boldsymbol{v}_r^{G_0} \cdot {}^N \boldsymbol{L}^G + {}^N \boldsymbol{\omega}_r^A \cdot {}^N \boldsymbol{H}^{SG/G_0} \tag{42} $$

$$ {}^N L_r^G = {}^N L_r^{SG} + I\left({}^N \boldsymbol{\omega}_r^B \cdot {}^N \boldsymbol{\omega}^B\right) \tag{43} $$

Note that, when one uses the motion variables in Eq. (26), the last term in Eq. (43) depends on r and simplifies to either $I\,{}^N\omega_i^B$ ($i = 1, 2, 3$) or 0.

5. *Generalized Effective/Inertia Force*

An efficient way to form ${}^N F_r^G$, the rth generalized effective force of a spherical gyrostat G in a reference frame N, is

$$ {}^N F_r^{RG} \triangleq {}^N \boldsymbol{v}_r^{G_0} \cdot {}^N \boldsymbol{F}^G + {}^N \boldsymbol{\omega}_r^A \cdot {}^N \boldsymbol{T}^{RG} \tag{44} $$

$$ {}^N F_r^G = {}^N F_r^{RG} + I\left[{}^N \boldsymbol{\omega}_r^A \cdot ({}^A \boldsymbol{\alpha}^B + {}^N \boldsymbol{\omega}^A \times {}^A \boldsymbol{\omega}^B) + {}^A \boldsymbol{\omega}_r^B \cdot {}^N \boldsymbol{\alpha}^B\right] \tag{45} $$

This first set of formulas is more efficient for fixed rotors than free rotors because, when ${}^A\boldsymbol{\omega}^B$ is prescribed (specified), ${}^A\boldsymbol{\omega}_r^B = \boldsymbol{0}$ and the last term in parentheses in Eq. (45) is 0. A second set of formulas for ${}^N F_r^G$, which is superior for free rotors, is

$$ {}^N F_r^{SG} \triangleq {}^N \boldsymbol{v}_r^{G_0} \cdot {}^N \boldsymbol{F}^G + {}^N \boldsymbol{\omega}_r^A \cdot {}^N \boldsymbol{T}^{SG} \tag{46} $$

$$ {}^N F_r^G = {}^N F_r^{SG} + I\,{}^N \boldsymbol{\omega}_r^B \cdot {}^N \boldsymbol{\alpha}^B \tag{47} $$

Note that, when one uses the motion variables in Eq. (26), the last term in Eq. (47) depends on r and simplifies to either $I\,{}^N\dot{\omega}_i^B$ ($i = 1, 2, 3$) or 0.

The generalized inertia force G in an implied reference frame N is denoted F_r^* and is defined

$$ F_r^* \triangleq -{}^N F_r^G \tag{48} $$

B. Spherical Gyrostat Example: Nutation Damper

Figure 2 is a schematic of a spherical gyrostat G consisting of a uniform sphere B and a carrier A. To characterize the mass distribution of G, one may introduce the constants I_{ij}^G, $i, j = 1, 2, 3$, the central inertia scalars of G for the unit vectors $\boldsymbol{a}_i$, $i = 1, 2, 3$, fixed in A; m^G, the mass of G; and I, the moment of inertia of B for any line passing through B_0. To describe the motion of G in the Newtonian reference frame N, the motion variables ω_i^A, ω_i^B, v_i, $i = 1, 2, 3$, are defined in accordance with Ref. 3 as

$$ \omega_i^A \triangleq {}^N \boldsymbol{\omega}^A \cdot \boldsymbol{a}_i \qquad (i = 1, 2, 3) $$

$$ \omega_i^B \triangleq {}^N \boldsymbol{\omega}^B \cdot \boldsymbol{a}_i \qquad (i = 1, 2, 3) $$

$$ v_i \triangleq {}^N \boldsymbol{v}^{G_0} \cdot \boldsymbol{a}_i \qquad (i = 1, 2, 3) $$

where ${}^N\boldsymbol{\omega}^A$ is the angular velocity of A in N, ${}^N\boldsymbol{\omega}^B$ is the angular velocity of B in N, and ${}^N\boldsymbol{v}^{G_0}$ is the velocity of G_0, the mass center of G, in N. The symbols F_r, $r = 1, \ldots, 9$, are generalized active forces arising from forces acting on A and B. With the aid of Eqs. (30), (39), and (47), expressions for the angular momentum, kinetic energy, and dynamic differential equations for $\dot{\omega}_i^A$, $\dot{\omega}_i^B$, and $\dot{v}_i$, $i = 1, 2, 3$, can be formed and are given as follows.

Angular momentum and kinetic energy:

$$ {}^N \boldsymbol{H}^{G/G_0} = \left(I_1^{SG}\omega_1^A + I\omega_1^B\right)\boldsymbol{a}_1 + \left(I_2^{SG}\omega_2^A + I\omega_2^B\right)\boldsymbol{a}_2 + \left(I_3^{SG}\omega_3^A + I\omega_3^B\right)\boldsymbol{a}_3 $$

$$ {}^N K^G = \tfrac{1}{2}\left[M^G\left(v_1^2 + v_2^2 + v_3^2\right) + I_1^{SG}{\omega_1^A}^2 + I_2^{SG}{\omega_2^A}^2 + I_3^{SG}{\omega_3^A}^2 + I\left({\omega_1^B}^2 + {\omega_2^B}^2 + {\omega_3^B}^2\right)\right] $$

Dynamical differential equations:

$$ \dot{\omega}_1^A = \left[\left(I_2^{SG} - I_3^{SG}\right)\omega_2^A\omega_3^A + F_1\right]\Big/ I_1^{SG} $$

$$ \dot{\omega}_2^A = \left[\left(I_3^{SG} - I_1^{SG}\right)\omega_1^A\omega_3^A + F_2\right]\Big/ I_2^{SG} $$

$$ \dot{\omega}_3^A = \left[\left(I_1^{SG} - I_2^{SG}\right)\omega_1^A\omega_2^A + F_3\right]\Big/ I_3^{SG} $$

$$ \dot{\omega}_1^B = \omega_3^A\omega_2^B - \omega_2^A\omega_3^B + F_4/I $$

$$ \dot{\omega}_2^B = \omega_1^A\omega_3^B - \omega_3^A\omega_1^B + F_5/I $$

$$ \dot{\omega}_3^B = \omega_2^A\omega_1^B - \omega_1^A\omega_2^B + F_6/I $$

$$ \dot{v}_1 = \omega_3^A v_2 - \omega_2^A v_3 + F_7\big/ M^G $$

$$ \dot{v}_2 = \omega_1^A v_3 - \omega_3^A v_1 + F_8\big/ M^G $$

$$ \dot{v}_3 = \omega_2^A v_1 - \omega_1^A v_2 + F_9\big/ M^G $$

The brevity of this set of equations is startling, especially when one realizes that when the mass distribution of G is treated in a traditional manner, that is, by introducing constants for the masses, mass center locations, and inertia scalars of A and B separately, then the traditional angular momentum expression is more than six times as long as its new counterpart, the traditional kinetic energy expression is more than twice as long as the one reported earlier, and the explicit equations governing $\dot{\omega}_i^A$, $\dot{\omega}_i^B$, and $\dot{v}_i$, $i = 1, 2, 3$, are more than four full pages long.

A second comparison between the traditional and new methods can be made by timing the following numerical motion simulation. A uniform sphere B of mass density 2 kg/m^3 and radius 2 m is placed in a spherical cavity of a rectangular parallelepiped A of mass density 4 kg/m^3 and dimension $10 \times 11 \times 7$ m. The center of the sphere is laterally displaced from the geometric center of the box by a distance of 1.4 m in the direction of the smallest dimension. The radius of the cavity is slightly larger than the radius of B, and the intervening space is filled with a viscous fluid so that one may assume that the set of forces acting on B (and in an opposite sense on A) can be replaced with an equivalent set, with its resultant placed at the B_0, and a couple whose torque is equal to $-c\,{}^A\boldsymbol{\omega}^B$, where $c = 536.2$ N·m·s.

To simulate 2000 s of motion with eight-digit accuracy, numerical integration of the dynamical differential equations were performed with a fourth-order Kutta–Merson (see Ref. 6, pages 24 and 25) variable time step integrator with a maximum integration step of 1.0 s on a 486-33-MHz computer. The initial values were $\omega_i^A = \omega_i^B = 3$ rad/s, $i = 1, 2, 3$, whereas those of v_i, $i = 1, 2, 3$, were such that the velocity of the geometric center of A in N was equal to zero. In their most computationally efficient form, the traditional set of equations required 203 s to integrate, whereas the new ones required only 79 s. The difference in integration time is directly attributable to the number of arithmetic operations that must be performed to evaluate $\dot{\omega}_i^A$, $\dot{\omega}_i^B$, and $\dot{v}_i$, $i = 1, 2, 3$, at each integration time step. When the traditional constants describe the mass distribution, 218 arithmetic operation are necessary, whereas only 43 arithmetic operations are needed when the new constants are employed. As one should expect, ${}^N K^G$, the kinetic energy of G in N, decreases during the integration, whereas ${}^N\boldsymbol{H}^{G/G_0}$, the central inertial angular momentum of G remains fixed in N.

One interesting result from the motion simulation can be inferred from Fig. 7, which shows the time history of the magnitude of ${}^A\boldsymbol{\omega}^B$. Initially, there is no angular motion of B relative to A, and hence, there is no dissipation of mechanical energy from the viscous fluid between A and B. However, this quasi-rigid motion is unstable, and subsequently the magnitude of ${}^A\boldsymbol{\omega}^B$ quickly increases, reaching a maximum of 0.7515 rad/s, 2.30 s later. After that, the magnitude of ${}^A\boldsymbol{\omega}^B$ decreases as B becomes entrained in A by viscous forces. In the limit as $t \to \infty$, A and B again become a quasi-rigid body, meaning that each point of B has the same velocity as the point of A with which it coincides. This quasi-rigid body has a flat spin in N. In other words, the quasi-rigid body has a simple angular velocity

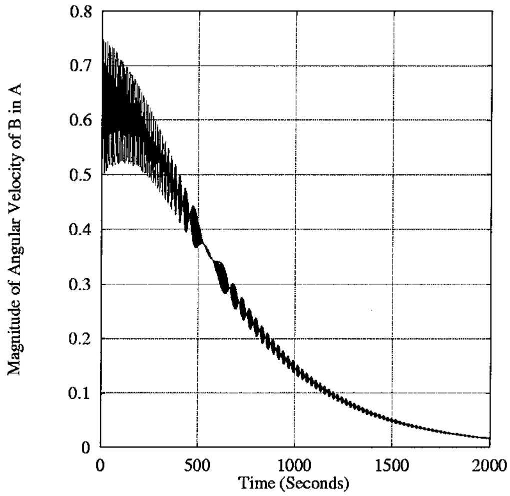

Fig. 7 Time History of the Magnitude of ${}^{A}\omega^{B}$.

in N, directed so that the maximum central principal axis of G is parallel to ${}^{N}\boldsymbol{H}^{G/G_0}$. The magnitude of the simple angular velocity is equal to $|{}^{N}\boldsymbol{H}^{G/G_0}|/I_{\max}$, where $I_{\max}$ is the maximum central moment of inertia of G.

V. Previous and New Work

Because of the broad range of applications in which gyrostats are used, it is a time-consuming task to examine even a small fraction of the previous literature written about the dynamic analysis of mechanical systems containing gyrostats. One difficult task undertaken by the authors was to discern whether any of the formulas for angular momentum, effective/inertia torque, kinetic energy, generalized momentum, and generalized effective/inertia force presented in Secs. III.A and IV.A have been previously established.

Of the 20 formulas for cylindrical gyrostats in Sec. III.A, 6 have been previously published in some form. The fixed-rotor formula for angular momentum of a cylindrical gyrostat provided in Eq. (4) appears in many places including Ref. 7, page 158 [Eq. (14)], Ref. 8, page 212 [Eq. (1)], and Ref. 9, page 67. The fixed-rotor formula for kinetic energy of a cylindrical gyrostat provided in Eq. (13) appears in Ref. 7, page 67 [Eq. (15)] and Ref. 8, page 212 [Eq. (3)], and the fixed-rotor formula for effective/inertia torque of a cylindrical gyrostat provided in Eq. (8) appears in Ref. 8, page 212 [Eq. (2)] and Ref. 9, page 67 [Eq. (4.54)]. Furthermore, Eqs. (6) and (15), the free-rotor formulas for the angular momentum and kinetic energy of a cylindrical gyrostat, appear in Ref. 7, page 178 [Eqs. (29) and (30)]. Certain case-specific formulas for calculating the generalized effective/inertia force of cylindrical gyrostats appear in Refs. 7 and 10, but those formulas differ from those presented in Sec. III.A and are highly restrictive because they are basis dependent and/or derived with a specific choice of motion variables. The basis dependence of those formulas detracts from the ability of vector symbolic manipulators to create highly efficient equations of motion for systems containing gyrostats. The restrictions to a specific choice of motion variables in those formulas limits the efficiency of equations of motion precluding the use of superior motion variables such as the ones proposed in Ref. 3.

On the other hand, many of the equations in Secs. III.A and IV.A appear to be new. To the best of the authors' knowledge, the applicability of Eqs. (4), (8), and (13), to spherical gyrostats is new because these equations do not appear in any of the aforementioned references on gyrostats[4,7–10] or any other prominent literature on gyrostats.[11–15] With the exception of Eqs. (6) and (15), all of the formulas in Secs. III.A and IV.A that are capable of employing the motion variables in Ref. 3 appear to be new. Last, the authors are fairly certain that all of the formulas for generalized momentum of a gyrostat [Eqs. (17), (19), (41), and (43)] and all of the formulas for the generalized effective/inertia force of gyrostats [Eqs. (21), (23), (45), and (47)] are new.

Derivations of the equations in Secs. III.A and IV.A are given in Appendix C.

VI. Conclusions

One should infer from the results of the three example problems that it can be highly efficacious to form equations of motion for gyrostats with the new formulas for angular momentum, effective/inertia torque, kinetic energy, generalized momentum, and generalized effective/inertia force. Although it is possible for certain simple systems to show that the new gyrostat formulas are equivalent to traditional formulas, it should be clear that by combining together the mass and inertia properties of the bodies that comprise a gyrostat, one can find shorter, more efficient, and less complicated expressions for angular momentum, effective/inertia torque, kinetic energy, generalized momentum, and generalized effective/inertia force.

It should be apparent from the three examples problems that, in treating a gyrostat in this new way, one needs to determine (experimentally or analytically) fewer mass distribution constants than when one treats a gyrostat in a traditional manner.

Appendix A: Traditional Helicopter Dynamical Differential Equation

$$\dot\omega_1 = -[(((b_2m^B + c_2m^C)(b_3m^B + c_3m^C)((b_3m^B+c_3m^C)(b_1m^B+c_1m^C)^2 - (m^A+m^B+m^C)((b_2m^B+c_2m^C)(I^A_{23} - b_2b_3m^B$$

$$- c_2c_3m^C) + (b_3m^B+c_3m^C)(I^A_{33}+J^B+m^B(b_1^2+b_2^2)+m^C(c_1^2+c_2^2))))) - (b_3m^B+c_3m^C)(b_1m^B+c_1m^C)^2((b_2m^B+c_2m^C)$$

$$\times (b_3m^B+c_3m^C) + (m^A+m^B+m^C)(I^A_{23} - b_2b_3m^B - c_2c_3m^C)) - (m^A+m^B+m^C)((b_2m^B+c_2m^C)(b_3m^B+c_3m^C)$$

$$+(m^A+m^B+m^C)(I^A_{23}-b_2b_3m^B-c_2c_3m^C))((b_1m^B+c_1m^C)(I^A_{13}-b_1b_3m^B-c_1c_3m^C)+(b_2m^B+c_2m^C)(I^A_{23}-b_2b_3m^B$$

$$-c_2c_3m^C)) - (m^A+m^B+m^C)((b_1m^B+c_1m^C)^2+(b_2m^B+c_2m^C)^2-(m^A+m^B+m^C)(I^A_{33}+J^B+m^B(b_1^2+b_2^2)$$

$$+m^C(c_1^2+c_2^2)))((b_1m^B+c_1m^C)(I^A_{12}-b_1b_2m^B-c_1c_2m^C)+(b_2m^B+c_2m^C)(I^A_{22}+J^C+m^B(b_1^2+b_3^2)+m^C(c_1^2+c_3^2))))$$

$$\times((m^A+m^B+m^C)\omega_1v_2 + (b_1m^B+c_1m^C)\omega_1\omega_3+(b_2m^B+c_2m^C)\omega_2\omega_3-(m^A+m^B+m^C)\omega_2v_1-(b_3m^B+c_3m^C)\omega_1^2$$

$$-(b_3m^B+c_3m^C)\omega_2^2 - F_8) + (m^A+m^B+m^C)((m^A+m^B+m^C)(b_1m^B+c_1m^C)(b_3m^B+c_3m^C)(I^A_{22}+J^C+m^B(b_1^2+b_3^2)$$

$$+m^C(c_1^2+c_3^2)) - (b_1m^B+c_1m^C)(b_3m^B+c_3m^C)^3-(b_3m^B+c_3m^C)(b_1m^B+c_1m^C)^3-(b_1m^B+c_1m^C)(b_3m^B+c_3m^C)$$

$$\times(b_2m^B+c_2m^C)^2-(m^A+m^B+m^C)(b_3m^B+c_3m^C)^2(I^A_{13}-b_1b_3m^B-c_1c_3m^C)-(m^A+m^B+m^C)(b_2m^B+c_2m^C)$$

$$\times(b_3m^B+c_3m^C)(I^A_{12}-b_1b_2m^B-c_1c_2m^C)-(m^A+m^B+m^C)(b_1m^B+c_1m^C)((b_1m^B+c_1m^C)(I^A_{13}-b_1b_3m^B-c_1c_3m^C)$$

$$+ (b_2 m^B + c_2 m^C)(I^A_{23} - b_2 b_3 m^B - c_2 c_3 m^C)) - (m^A + m^B + m^C)^2((I^A_{12} - b_1 b_2 m^B - c_1 c_2 m^C)(I^A_{23} - b_2 b_3 m^B - c_2 c_3 m^C)$$

$$- (I^A_{13} - b_1 b_3 m^B - c_1 c_3 m^C)(I^A_{22} + J^C + m^B(b_1^2 + b_3^2) + m^C(c_1^2 + c_3^2)))))(I^B \omega_1 \omega^B + (b_1 m^B + c_1 m^C)\omega_3 v_1 + (b_2 m^B + c_2 m^C)\omega_3 v_2$$

$$+ (I^A_{12} - b_1 b_2 m^B - c_1 c_2 m^C)\omega_1^2 + (I^A_{23} - b_2 b_3 m^B - c_2 c_3 m^C)\omega_1 \omega_3 + (I^A_{22} + m^B b_1^2 + m^C c_1^2 - I^A_{11} - J^B - m^B b_2^2 - m^C c_2^2)\omega_1 \omega_2$$

$$- (b_1 m^B + c_1 m^C)\omega_1 v_3 - (b_2 m^B + c_2 m^C)\omega_2 v_3 - (I^A_{12} - b_1 b_2 m^B - c_1 c_2 m^C)\omega_2^2 - (I^A_{13} - b_1 b_3 m^B - c_1 c_3 m^C)\omega_2 \omega_3 - F_3)$$

$$+ (m^A + m^B + m^C)((m^A + m^B + m^C)(b_1 m^B + c_1 m^C)(b_2 m^B + c_2 m^C)(I^A_{33} + J^B + m^B(b_1^2 + b_2^2) + m^C(c_1^2 + c_2^2)) - (b_1 m^B$$

$$+ c_1 m^C)(b_2 m^B + c_2 m^C)^3 - (b_2 m^B + c_2 m^C)(b_1 m^B + c_1 m^C)^3 - (b_1 m^B + c_1 m^C)(b_2 m^B + c_2 m^C)(b_3 m^B + c_3 m^C)^2 - (m^A + m^B$$

$$+ m^C)(b_2 m^B + c_2 m^C)^2(I^A_{12} - b_1 b_2 m^B - c_1 c_2 m^C) - (m^A + m^B + m^C)(b_2 m^B + c_2 m^C)(b_3 m^B + c_3 m^C)(I^A_{13} - b_1 b_3 m^B - c_1 c_3 m^C)$$

$$- (m^A + m^B + m^C)(b_1 m^B + c_1 m^C)((b_1 m^B + c_1 m^C)(I^A_{12} - b_1 b_2 m^B - c_1 c_2 m^C) + (b_3 m^B + c_3 m^C)(I^A_{23} - b_2 b_3 m^B - c_2 c_3 m^C))$$

$$- (m^A + m^B + m^C)^2((I^A_{13} - b_1 b_3 m^B - c_1 c_3 m^C)(I^A_{23} - b_2 b_3 m^B - c_2 c_3 m^C) - (I^A_{12} - b_1 b_2 m^B - c_1 c_2 m^C)(I^A_{33} + J^B + m^B(b_1^2 + b_2^2)$$

$$+ m^C(c_1^2 + c_2^2))))((b_1 m^B + c_1 m^C)\omega_2 v_1 + (b_3 m^B + c_3 m^C)\omega_2 v_3 + (I^A_{12} - b_1 b_2 m^B - c_1 c_2 m^C)\omega_2 \omega_3 + (I^A_{13} - b_1 b_3 m^B - c_1 c_3 m^C)\omega_3^2$$

$$- I^C \omega_1 \omega^C - (b_1 m^B + c_1 m^C)\omega_1 v_2 - (b_3 m^B + c_3 m^C)\omega_3 v_2 - (I^A_{13} - b_1 b_3 m^B - c_1 c_3 m^C)\omega_1^2 - (I^A_{23} - b_2 b_3 m^B - c_2 c_3 m^C)\omega_1 \omega_2$$

$$- (I^A_{33} + m^B b_1^2 + m^C c_1^2 - I^A_{11} - J^C - m^B b_3^2 - m^C c_3^2)\omega_1 \omega_3 - F_2) + ((m^A + m^B + m^C)(b_1 m^B + c_1 m^C)(b_3 m^B + c_3 m^C)^2$$

$$\times (I^A_{23} - b_2 b_3 m^B - c_2 c_3 m^C) + (b_3 m^B + c_3 m^C)(m^A + m^B + m^C)^2(I^A_{13} - b_1 b_3 m^B - c_1 c_3 m^C)(I^A_{23} - b_2 b_3 m^B - c_2 c_3 m^C)$$

$$+ (m^A + m^B + m^C)(b_1 m^B + c_1 m^C)(b_2 m^B + c_2 m^C) + (b_3 m^B + c_3 m^C)(I^A_{22} + J^C + m^B(b_1^2 + b_3^2) + m^C(c_1^2 + c_3^2))) + (b_1 m^B + c_1 m^C)$$

$$\times (b_2 m^B + c_2 m^C)(b_3 m^B + c_3 m^C)((b_1 m^B + c_1 m^C)^2 - (m^A + m^B + m^C)(I^A_{33} + J^B + m^B(b_1^2 + b_2^2) + m^C(c_1^2 + c_2^2))) + (m^A + m^B$$

$$+ m^C)(b_3 m^B + c_3 m^C)(I^A_{12} - b_1 b_2 m^B - c_1 c_2 m^C)((b_1 m^B + c_1 m^C)^2 - (m^A + m^B + m^C)(I^A_{33} + J^B + m^B(b_1^2 + b_2^2) + m^C(c_1^2 + c_2^2)))$$

$$- (b_2 m^B + c_2 m^C)(b_3 m^B + c_3 m^C)(b_1 m^B + c_1 m^C)^3 - (m^A + m^B + m^C)(b_1 m^B + c_1 m^C)(b_2 m^B + c_2 m^C)$$

$$\times ((b_1 m^B + c_1 m^C)(I^A_{13} - b_1 b_3 m^B - c_1 c_3 m^C) + \cdots$$

Appendix B: New Helicopter Dynamical Differential Equations

$$\dot{\omega}_1 = [\{I^G_{12} I^G_{23} + I^G_{13}(I^B - I^G_{22})\}\{I^B \omega_1 \omega^B + I^G_{12}\omega_1^2 + I^G_{23}\omega_1 \omega_3 - I^G_{12}\omega_2^2 - I^G_{13}\omega_2 \omega_3 - (I^B + I^G_{11} - I^G_{22})\omega_1 \omega_2 - F_3\} + \{I^G_{13} I^G_{23} + I^G_{12}(I^C - I^G_{33})\}$$

$$\times \{I^G_{12}\omega_2 \omega_3 + I^G_{13}\omega_3^2 + (I^C + I^G_{11} - I^G_{33})\omega_1 \omega_3 - I^C \omega_1 \omega^C - I^G_{13}\omega_1^2 - I^G_{23}\omega_1 \omega_2 - F_2\} - \{I^{G\,2}_{23} - (I^B - I^G_{22})(I^C - I^G_{33})\}\{I^C \omega_2 \omega^C$$

$$+ I^G_{13}\omega_1 \omega_2 + I^G_{23}\omega_2^2 - I^B \omega_3 \omega^B - I^G_{12}\omega_1 \omega_3 - I^G_{23}\omega_3^2 - (I^C + I^G_{22} - I^B - I^G_{33})\omega_2 \omega_3 - F_1\}]/[I^G_{23}(I^G_{11} I^G_{23} - 2I^G_{12} I^G_{13}) - I^{G\,2}_{13}(I^B - I^G_{22})$$

$$- (I^C - I^G_{33})\{I^{G\,2}_{12} + I^G_{11}(I^B - I^G_{22})\}]$$

$$\dot{\omega}_2 = [(I^G_{13} I^G_{23} + I^G_{12}(I^C - I^G_{33}))(I^C \omega_2 \omega^C + I^G_{13}\omega_1 \omega_2 + I^G_{23}\omega_2^2 - I^B \omega_3 \omega^B - I^G_{12}\omega_1 \omega_3 - I^G_{23}\omega_3^2 - (I^C + I^G_{22} - I^B - I^G_{33})\omega_2 \omega_3 - F_1)$$

$$- (I^G_{11} I^G_{23} - I^G_{12} I^G_{13})(I^B \omega_1 \omega^B + I^G_{12}\omega_1^2 + I^G_{23}\omega_1 \omega_3 - I^G_{12}\omega_2^2 - I^G_{13}\omega_2 \omega_3 - (I^B + I^G_{11} - I^G_{22})\omega_1 \omega_2 - F_3) - (I^{G\,2}_{13} + I^G_{11}(I^C - I^G_{33}))$$

$$\times (I^G_{12}\omega_2 \omega_3 + I^G_{13}\omega_3^2 + (I^C + I^G_{11} - I^G_{33})\omega_1 \omega_3 - I^C \omega_1 \omega^C - I^G_{13}\omega_1^2 - I^G_{23}\omega_1 \omega_2 - F_2)]/[I^G_{23}(I^G_{11} I^G_{23} - 2I^G_{12} I^G_{13}) - I^{G\,2}_{13}(I^B - I^G_{22})$$

$$- (I^C - I^G_{33})(I^{G\,2}_{12} + I^G_{11}(I^B - I^G_{22}))]$$

$$\dot{\omega}_3 = [(I^G_{12} I^G_{23} + I^G_{13}(I^B - I^G_{22}))(I^C \omega_2 \omega^C + I^G_{13}\omega_1 \omega_2 + I^G_{23}\omega_2^2 - I^B \omega_3 \omega^B - I^G_{12}\omega_1 \omega_3 - I^G_{23}\omega_3^2 - (I^C + I^G_{22} - I^B - I^G_{33})\omega_2 \omega_3 - F_1)$$

$$- (I^G_{11} I^G_{23} - I^G_{12} I^G_{13})(I^G_{12}\omega_2 \omega_3 + I^G_{13}\omega_3^2 + (I^C + I^G_{11} - I^G_{33})\omega_1 \omega_3 - I^C \omega_1 \omega^C - I^G_{13}\omega_1^2 - I^G_{23}\omega_1 \omega_2 - F_2) - (I^{G\,2}_{12} + I^G_{11}(I^B - I^G_{22}))$$

$$\times (I^B \omega_1 \omega^B + I^G_{12}\omega_1^2 + I^G_{23}\omega_1 \omega_3 - I^G_{12}\omega_2^2 - I^G_{13}\omega_2 \omega_3 - (I^B + I^G_{11} - I^G_{22})\omega_1 \omega_2 - F_3)]/[I^G_{23}(I^G_{11} I^G_{23} - 2I^G_{12} I^G_{13}) - I^{G\,2}_{13}(I^B - I^G_{22})$$

$$- (I^C - I^G_{33})(I^{G\,2}_{12} + I^G_{11}(I^B - I^G_{22}))]$$

$$\dot{\omega}^B = F_4/I^B, \qquad \dot{\omega}^C = F_5/I^C, \qquad \dot{v}_1 = \omega_3 v_2 - \omega_2 v_3 + F_6/m^G, \qquad \dot{v}_2 = \omega_1 v_3 - \omega_3 v_1 + F_7/m^G, \qquad \dot{v}_3 = \omega_2 v_1 - \omega_1 v_2 + F_8/m^G$$

Appendix C: Derivations

The derivation of each formula in Secs. III.A and IV.A is presented in this section. Each gyrostat is named G and consists of a carrier A and a rotor B, both of which move in a reference frame N.

There are several kinematical and mass distribution relationships that are repeatedly used in the derivations that follow. For example, to express certain dynamic quantities in terms of parameters that describe the mass distribution and motion of G in N, it proves analytically useful to introduce a rigid body RG that has the same mass distribution as G, but the same motion as A. Thus, the mass of RG is equivalent to m^G (the mass of G), and the central inertia dyadic of RG is equal to $\underline{\boldsymbol{I}}^G$ (the central inertia dyadic of G).

The general kinematic relationships are as follows. The angular velocity addition theorem [see Ref. 5, page 24, Eq. (2.4.1)] enables one to write

$$ {}^N\boldsymbol{\omega}^B = {}^N\boldsymbol{\omega}^A + {}^A\boldsymbol{\omega}^B \tag{C1} $$

$$ {}^N\boldsymbol{\omega}_r{}^B \underset{(\text{C1})}{=} {}^N\boldsymbol{\omega}_r^A + {}^A\boldsymbol{\omega}_r^B \tag{C2} $$

$$ {}^A\boldsymbol{\omega}^B \underset{(\text{C1})}{=} {}^N\boldsymbol{\omega}^B - {}^N\boldsymbol{\omega}^A \tag{C3} $$

The velocity of B_i, a generic particle of B, in N can be written [see Ref. 5, page 32, Eq. (2.8.1)] as the sum of the velocity, in N, of AB_i, the point of A that is coincident with B_i at the instant under consideration, and the velocity of B_i in A as

$$ {}^N\boldsymbol{v}^{B_i} = {}^N\boldsymbol{v}^{AB_i} + {}^A\boldsymbol{v}^{B_i} \tag{C4} $$

Because B_0 (the mass center of B) is fixed in A, ${}^A\boldsymbol{v}^{B_i}$ can be expressed in terms of ${}^A\boldsymbol{\omega}^B$ (the angular velocity of B in A) and $\boldsymbol{r}_i$ (the position vector of B_i from B_0) as

$$ {}^A\boldsymbol{v}^{B_i} = {}^A\boldsymbol{\omega}^B \times \boldsymbol{r}_i \tag{C5} $$

The general mass distribution relationships are as follows. From B_0 being the mass center of B it follows that

$$ \sum_{i=1}^{\beta} m^{B_i}\boldsymbol{r}_i = 0 \tag{C6} $$

and, on twice differentiating Eq. (C6) with respect to time in A, one concludes that

$$ \sum_{i=1}^{\beta} m^{B_i}\,{}^A\boldsymbol{v}^{B_i} \underset{(\text{C6})}{=} 0 \tag{C7} $$

$$ \sum_{i=1}^{\beta} m^{B_i}\,{}^A\boldsymbol{a}^{B_i} \underset{(\text{C7})}{=} 0 \tag{C8} $$

Useful relationships for proofs of cylindrical gyrostat formulas are given next. When the discussion is restricted to cylindrical gyrostats, ${}^A\boldsymbol{\omega}^B$ can be written in terms of an angular speed Ω and a unit vector $\boldsymbol{b}$ fixed in A and parallel to the symmetry axis of B as

$$ {}^A\boldsymbol{\omega}^B = \Omega\boldsymbol{b} \tag{C9} $$

whereas $\underline{\boldsymbol{I}}^B$ can be specified in terms of J (the moment of inertia of B about the symmetry axis), K (the moment of inertia of B about any line perpendicular to $\boldsymbol{b}$ and passing through the mass center of B), and $\underline{\mathbf{1}}$ (the unit dyadic) as

$$ \underline{\boldsymbol{I}}^B = K\underline{\mathbf{1}} + (J-K)\boldsymbol{bb} \tag{C10} $$

One immediate consequence of Eq. (C10) is that, for any vector $\boldsymbol{v}$ parallel to $\boldsymbol{b}$,

$$ \boldsymbol{v}\cdot\underline{\boldsymbol{I}}^B \underset{(\text{C10})}{=} \boldsymbol{v}\cdot J\boldsymbol{bb} = J\boldsymbol{v} $$

which means that

$$ {}^A\boldsymbol{\omega}^B\cdot\underline{\boldsymbol{I}}^B \underset{(\text{C10})}{=} {}^A\boldsymbol{\omega}^B\cdot J\boldsymbol{bb} \underset{(\text{C9})}{=} J\,{}^A\boldsymbol{\omega}^B \tag{C11} $$

Another mass distribution quantity that helps simplify dynamic expressions for cylindrical gyrostats is the dyadic $\underline{\boldsymbol{I}}^{CG}\cdot\underline{\boldsymbol{I}}^{CG}$ is related to $\underline{\boldsymbol{I}}^G$ by

$$ \underline{\boldsymbol{I}}^{CG} \triangleq \underline{\boldsymbol{I}}^G - J\boldsymbol{bb} \tag{C12} $$

Useful relationships for proofs of spherical gyrostat formulas are given as follows. When the discussion is restricted to spherical gyrostats, note that $\underline{\boldsymbol{I}}^B$ can be expressed in terms of I (the moment of inertia of B about any line passing through B_0) and $\underline{\mathbf{1}}$ (the unit dyadic) as

$$ \underline{\boldsymbol{I}}^B = I\underline{\mathbf{1}} \tag{C13} $$

One immediate consequence of Eq. (C13) is that, for any vector $\boldsymbol{v}$,

$$ \boldsymbol{v}\cdot\underline{\boldsymbol{I}}^B \underset{(\text{C13})}{=} I\boldsymbol{v}\cdot\underline{\mathbf{1}} = I\boldsymbol{v} $$

which means that

$$ {}^A\boldsymbol{\omega}^B\cdot\underline{\boldsymbol{I}}^B \underset{(\text{C13})}{=} I\,{}^A\boldsymbol{\omega}^B\cdot\underline{\mathbf{1}} = I\,{}^A\boldsymbol{\omega}^B \tag{C14} $$

Another mass distribution quantity that helps simplify dynamic expressions for spherical gyrostats is the dyadic $\underline{\boldsymbol{I}}^{SG}\cdot\underline{\boldsymbol{I}}^{SG}$ is related to $\underline{\boldsymbol{I}}^G$ by

$$ \underline{\boldsymbol{I}}^{SG} \triangleq \underline{\boldsymbol{I}}^G - \underline{\boldsymbol{I}}^B \underset{(\text{C13})}{=} \underline{\boldsymbol{I}}^G - I\underline{\mathbf{1}} \tag{C15} $$

A. Angular Momentum

To establish the validity of Eqs. (4), (6), (28), and (30), start by noting that the central angular momentum of G in N of a set of α particles $A_1,\ldots,A_\alpha$ of a rigid body A and a set of β particles $B_1,\ldots,B_\beta$ of a rigid body B is defined (see Ref. 5, page 69) in terms of $\boldsymbol{p}^{G_0A_i}$ (the position vector of A_i from G_0, the mass center of G), ${}^N\boldsymbol{L}^{A_i}$ (the linear momentum of A_i in N), $\boldsymbol{p}^{G_0B_i}$ (the position vector of B_i from G_0), and ${}^N\boldsymbol{L}^{B_i}$ (the linear momentum of A_i in N) as

$$ {}^N\boldsymbol{H}^{G/G_0} \triangleq \sum_{i=1}^{\alpha}\boldsymbol{p}^{G_0A_i}\times{}^N\boldsymbol{L}^{A_i} + \sum_{i=1}^{\beta}\boldsymbol{p}^{G_0B_i}\times{}^N\boldsymbol{L}^{B_i} \tag{C16} $$

Because ${}^N\boldsymbol{L}^{A_i}$ and ${}^N\boldsymbol{L}^{B_i}$ are defined in terms of m^{A_i} (the mass of A_i), ${}^N\boldsymbol{L}^{A_i}$ (the velocity of A_i in N), m^{B_i} (the mass of B_i), and ${}^N\boldsymbol{v}^{B_i}$ (the velocity of B_i in N) as

$$ {}^N\boldsymbol{L}^{A_i} \triangleq m^{A_i}\,{}^N\boldsymbol{v}^{A_i} \tag{C17} $$

$$ {}^N\boldsymbol{L}^{B_i} \triangleq m^{B_i}\,{}^N\boldsymbol{v}^{B_i} \tag{C18} $$

Eq. (C16) can be rewritten as

$$ {}^N\boldsymbol{H}^{G/G_0} \underset{(\text{C16–C18})}{=} \sum_{i=1}^{\alpha} m^{A_i}\boldsymbol{p}^{G_0A_i}\times{}^N\boldsymbol{v}^{A_i} + \sum_{i=1}^{\beta} m^{B_i}\boldsymbol{p}^{G_0B_i}\times{}^N\boldsymbol{v}^{B_i} \tag{C19} $$

Replacing ${}^N\boldsymbol{v}^{B_i}$ in Eq. (C19) with the right-hand side of Eq. (C4) gives

$$ {}^N\boldsymbol{H}^{G/G_0} \underset{(\text{C19})(\text{C4})}{=} \sum_{i=1}^{\alpha} m^{A_i}\boldsymbol{p}^{G_0A_i}\times{}^N\boldsymbol{v}^{A_i} + \sum_{i=1}^{\beta} m^{B_i}\boldsymbol{p}^{G_0B_i}\times{}^N\boldsymbol{v}^{AB_i} + \sum_{i=1}^{\beta} m^{B_i}\boldsymbol{p}^{G_0B_i}\times{}^A\boldsymbol{v}^{B_i} \tag{C20} $$

The first two terms in Eq. (C20) are, by definition, ${}^N\boldsymbol{H}^{RG/G_0}$. A derivation of the more practical formula [Eq. (3)] for calculating ${}^N\boldsymbol{H}^{RG/G_0}$ is found in Ref. 5, pages 69 and 70. Equation (C20) can be further simplified by rewriting $\boldsymbol{p}^{G_0B_i}$ as the sum of $\boldsymbol{p}^{G_0B_0}$ (the position vector of B_0 from G_0) and $\boldsymbol{r}_i$ (the position vector of B_i from B_0) as

$$ \boldsymbol{p}^{G_0B_i} = \boldsymbol{p}^{G_0B_0} + \boldsymbol{r}_i \tag{C21} $$

which results in

$$ {}^{N}\boldsymbol{H}^{G/G_0} \underset{(C20)(C21)}{=} {}^{N}\boldsymbol{H}^{RG/G_0} + \boldsymbol{p}^{G_0 B_0} \times \sum_{i=1}^{\beta} m^{B_i}\, {}^{A}\boldsymbol{v}^{B_i} + \sum_{i=1}^{\beta} m^{B_i} \boldsymbol{r}_i \times {}^{A}\boldsymbol{v}^{B_i} \tag{C22} $$

The second term in Eq. (C22) is **0** [see Eq. (C7)] and the third term can be reexpressed by replacing ${}^{A}\boldsymbol{v}^{B_i}$ with the right-hand side of Eq. (C5) producing

$$ {}^{N}\boldsymbol{H}^{G/G_0} \underset{(C5)(C22)}{=} {}^{N}\boldsymbol{H}^{RG/G_0} + \sum_{i=1}^{\beta} m^{B_i} \boldsymbol{r}_i \times \left({}^{A}\boldsymbol{\omega}^{B} \times \boldsymbol{r}_i\right) \tag{C23} $$

In light of the derivation in Ref. 5, page 70, Eq. (C23) can be simplified to

$$ {}^{N}\boldsymbol{H}^{G/G_0} \underset{(C23)}{=} {}^{N}\boldsymbol{H}^{RG/G_0} + \underline{\boldsymbol{I}}^{B} \cdot {}^{A}\boldsymbol{\omega}^{B} \tag{C24} $$

Equation (4), the formula for the angular momentum of a cylindrical gyrostat, results directly from substitution of Eq. (C11) into Eq. (C24). After substitution from Eq. (C13) into Eq. (C24), it is easily seen that Eq. (4) applies to spherical gyrostats once J is replaced with I.

Substitution of Eqs. (C11) and (3) into Eq. (C24) leads to

$$ {}^{N}\boldsymbol{H}^{G/G_0} = \underset{(3)}{\underline{\boldsymbol{I}}^{G} \cdot {}^{N}\boldsymbol{\omega}^{A}} + \underset{(C24)}{J\boldsymbol{bb} \cdot {}^{A}\boldsymbol{\omega}^{B}} \tag{C25} $$

Replacing ${}^{A}\boldsymbol{\omega}^{B}$ with its right-hand side [see Eq. (C3)] and subsequent rearrangement and substitution of Eq. (C12) gives

$$ \begin{aligned} {}^{N}\boldsymbol{H}^{G/G_0} &\underset{(C25)}{=} \underline{\boldsymbol{I}}^{G} \cdot {}^{N}\boldsymbol{\omega}^{A} + J\boldsymbol{bb} \cdot \underset{(51)}{({}^{N}\boldsymbol{\omega}^{B} - {}^{N}\boldsymbol{\omega}^{A})} \\ &= (\underline{\boldsymbol{I}}^{G} - J\boldsymbol{bb}) \cdot {}^{N}\boldsymbol{\omega}^{A} + J\boldsymbol{bb} \cdot {}^{N}\boldsymbol{\omega}^{B} \\ &= \underset{(C12)}{\underline{\boldsymbol{I}}^{CG}} \cdot {}^{N}\boldsymbol{\omega}^{A} + J\boldsymbol{bb} \cdot {}^{N}\boldsymbol{\omega}^{B} \\ &= \underset{(5)}{{}^{N}\boldsymbol{H}^{CG/G_0}} + J\boldsymbol{bb} \cdot {}^{N}\boldsymbol{\omega}^{B} \end{aligned} \tag{C26} $$

which results in Eq. (6), the formula for the angular momentum of a cylindrical gyrostat.

Equation (30), the formula for the angular momentum of a spherical gyrostat, is deduced in a similar fashion. The derivation is shown without comment:

$$ \begin{aligned} {}^{N}\boldsymbol{H}^{G/G_0} &\underset{(C24)}{=} \underset{(3)}{\underline{\boldsymbol{I}}^{G} \cdot {}^{N}\boldsymbol{\omega}^{A}} + I\underline{\boldsymbol{1}} \cdot \underset{(C2)}{({}^{N}\boldsymbol{\omega}^{B} - {}^{N}\boldsymbol{\omega}^{A})} \\ &= (\underline{\boldsymbol{I}}^{G} - I\underline{\boldsymbol{1}}) \cdot {}^{N}\boldsymbol{\omega}^{A} + I\,{}^{N}\boldsymbol{\omega}^{B} \\ &= \underset{(C15)}{\underline{\boldsymbol{I}}^{SG} \cdot {}^{N}\boldsymbol{\omega}^{A}} + I\,{}^{N}\boldsymbol{\omega}^{B} \\ &= \underset{(29)}{{}^{N}\boldsymbol{H}^{SG/G_0}} + I\,{}^{N}\boldsymbol{\omega}^{B} \end{aligned} \tag{C27} $$

B. Effective and Inertia Torque

To establish the validity of Eqs. (8), (10), (32), and (34), start by noting that the effective torque of any set of ν particles $P_1, \ldots, P_\nu$ moving in a reference frame N is defined in terms of $\boldsymbol{p}^{S_0 P_i}$ (the position vector of P_i from S_0, the mass center of S) and ${}^{N}\boldsymbol{F}^{P_i}$ (the effective force of P_i in N) as

$$ {}^{N}\boldsymbol{T}^{S} \triangleq \sum_{i=1}^{\nu} \boldsymbol{p}^{S_0 P_i} \times {}^{N}\boldsymbol{F}^{P_i} \tag{C28} $$

which is equal to the moment of the effective forces about S_0. Because ${}^{N}\boldsymbol{F}^{P_i}$ is defined in terms of m^{P_i} (the mass of P_i) and ${}^{N}\boldsymbol{a}^{P_i}$ (the acceleration of P_i in N) as

$$ {}^{N}\boldsymbol{F}^{P_i} \triangleq m^{P_i}\, {}^{N}\boldsymbol{a}^{P_i} \tag{C29} $$

Eq. (C28) can be rewritten as

$$ {}^{N}\boldsymbol{T}^{S} \underset{(C28)(C29)}{=} \sum_{i=1}^{\nu} m^{P_i} \boldsymbol{p}^{S_0 P_i} \times {}^{N}\boldsymbol{a}^{P_i} \tag{C30} $$

With some minor manipulation, it is easily shown that the effective torque of S is related to the time rate of change of the central angular momentum of S by

$$ {}^{N}\boldsymbol{T}^{S} = \frac{{}^{N}\mathrm{d}\,{}^{N}\boldsymbol{H}^{S/S_0}}{\mathrm{d}t} \tag{C31} $$

With Eq. (31) in hand, the formulas for ${}^{N}\boldsymbol{T}^{RG}$ and ${}^{N}\boldsymbol{T}^{G}$ are deduced most easily by differentiating the corresponding equations for ${}^{N}\boldsymbol{H}^{RG/G_0}$ and ${}^{N}\boldsymbol{H}^{G/G_0}$. For example, Eq. (7), the formula for ${}^{N}\boldsymbol{T}^{RG}$, is found by substituting Eq. (3) into Eq. (C31) to arrive at

$$ {}^{N}\boldsymbol{T}^{RG} \underset{(C31)}{=} \frac{{}^{N}\mathrm{d}\,{}^{N}\boldsymbol{H}^{RG/G_0}}{\mathrm{d}t} \tag{C32} $$

$$ = \frac{{}^{A}\mathrm{d}\,{}^{N}\boldsymbol{H}^{RG/G_0}}{\mathrm{d}t} + {}^{N}\boldsymbol{\omega}^{A} \times {}^{N}\boldsymbol{H}^{RG/G_0} \tag{C33} $$

$$ \underset{(3)}{=} \underline{\boldsymbol{I}}^{G} \cdot {}^{N}\boldsymbol{\alpha}^{A} + {}^{N}\boldsymbol{\omega}^{A} \times \underline{\boldsymbol{I}}^{G} \cdot {}^{N}\boldsymbol{\omega}^{A} \tag{C34} $$

Similarly, Eq. (8), the formula for the effective torque of cylindrical gyrostats, can be found by substituting Eq. (4) into Eq. (C31) and subsequently making use of Eq. (32) to produce

$$ \begin{aligned} {}^{N}\boldsymbol{T}^{G} &\underset{(C31)}{=} \frac{{}^{N}\mathrm{d}\,{}^{N}\boldsymbol{H}^{G/G_0}}{\mathrm{d}t} \\ &\underset{(4)}{=} \frac{{}^{N}\mathrm{d}\,{}^{N}\boldsymbol{H}^{RG/G_0}}{\mathrm{d}t} + \frac{{}^{N}\mathrm{d}(J\,{}^{A}\boldsymbol{\omega}^{B})}{\mathrm{d}t} \\ &= \underset{(C32)}{{}^{N}\boldsymbol{T}^{RG}} + J({}^{A}\boldsymbol{\alpha}^{B} + {}^{N}\boldsymbol{\omega}^{A} \times {}^{A}\boldsymbol{\omega}^{B}) \end{aligned} \tag{C35} $$

C. Kinetic Energy

To establish the validity of Eqs. (13), (15), (37), and (39), start by noting that the kinetic energy of G in N of a set of α particles $A_1, \ldots, A_\alpha$ of A and a set of β particles $B_1, \ldots, B_\beta$ of a B is defined [see Ref. 5, page 147, Eq. (5.4.1)] in terms of m^{A_i} and m^{B_i} (the masses of A_i and B_i, respectively) and ${}^{N}\boldsymbol{v}^{A_i}$ and ${}^{N}\boldsymbol{v}^{B_i}$ (the velocities of A_i and B_i in N) as

$$ {}^{N}K^{G} \triangleq \frac{1}{2}\sum_{i=1}^{\alpha} m^{A_i}\, {}^{N}\boldsymbol{v}^{A_i} \cdot {}^{N}\boldsymbol{v}^{A_i} + \frac{1}{2}\sum_{i=1}^{\beta} m^{B_i}\, {}^{N}\boldsymbol{v}^{B_i} \cdot {}^{N}\boldsymbol{v}^{B_i} \tag{C36} $$

The velocity of B_i in N in Eq. (C36) can be reexpressed with the assistance of Eq. (C4), so that

$$ {}^{N}K^{G} = \frac{1}{2}\sum_{i=1}^{\alpha} m^{A_i}\, {}^{N}\boldsymbol{v}^{A_i^2} + \frac{1}{2}\sum_{i=1}^{\beta} m^{B_i}\, {}^{N}\boldsymbol{v}^{AB_i^2} + \frac{1}{2}\sum_{i=1}^{\beta} m^{B_i}\, {}^{A}\boldsymbol{v}^{B_i^2} + \sum_{i=1}^{\beta} m^{B_i}\, {}^{N}\boldsymbol{v}^{AB_i} \cdot {}^{A}\boldsymbol{v}^{B_i} \tag{C37} $$

By definition, ${}^{N}K^{RG}$ is the sum of the first two terms in Eq. (C37). A derivation of the more practical formula [Eq. (12)] for calculating ${}^{N}K^{RG}$ is found in Ref. 5, page 148. With the aid of Eq. (C5) and the derivation found in Ref. 5, page 148, it can be shown that the third term in Eq. (C37) is

$$ \begin{aligned} \frac{1}{2}\sum_{i=1}^{\beta} m^{B_i}\, {}^{A}\boldsymbol{v}^{B_i^2} &\underset{(C5)}{=} \frac{1}{2}\sum_{i=1}^{\beta} m^{B_i} \left({}^{A}\boldsymbol{\omega}^{B} \times \boldsymbol{r}_i\right)^2 \\ &= \frac{1}{2}\,{}^{A}\boldsymbol{\omega}^{B} \cdot \underline{\boldsymbol{I}}^{B} \cdot {}^{A}\boldsymbol{\omega}^{B} \end{aligned} \tag{C38} $$

The last term in Eq. (C37) can be reformed by using an expression for ${}^{N}\boldsymbol{v}^{AB_i}$, which follows from that both AB_i and B_0 are fixed on A, namely,

$$ {}^{N}\boldsymbol{v}^{AB_i} = {}^{N}\boldsymbol{v}^{B_0} + {}^{N}\boldsymbol{\omega}^{A} \times \boldsymbol{r}_i \tag{C39} $$

Subsequent use of Eqs. (C5) and (C7) and the formula found in Ref. 5, page 70 [Eq. (3.6.31)] in Eq. (C37) enables one to write

$$\sum_{i=1}^{\beta} m^{B_i}\,{}^{N}\boldsymbol{v}^{AB_i}\cdot{}^{A}\boldsymbol{v}^{B_i} \underset{(C39)}{=} {}^{N}\boldsymbol{v}^{B_0}\cdot\sum_{i=1}^{\beta} m^{B_i}\,{}^{A}\boldsymbol{v}^{B_i} + {}^{N}\boldsymbol{\omega}^{A}\times\sum_{i=1}^{\beta} m^{B_i}\boldsymbol{r}_i\cdot{}^{A}\boldsymbol{v}^{B_i}$$

$$= \underset{(C7)}{0} + {}^{N}\boldsymbol{\omega}^{A}\cdot\sum_{i=1}^{\beta} m^{B_i}\boldsymbol{r}_i\times\underset{(C5)}{\left({}^{A}\boldsymbol{\omega}^{B}\times\boldsymbol{r}_i\right)}$$

$$= {}^{N}\boldsymbol{\omega}^{A}\cdot\underline{\boldsymbol{I}}^{B}\cdot{}^{A}\boldsymbol{\omega}^{B} \tag{C40}$$

Collecting the information from Eqs. (C37), (C38), and (C40) allows one to write

$${}^{N}K^{G} \underset{(C37)}{=} {}^{N}K^{RG} + \underset{(C38)}{\tfrac{1}{2}{}^{A}\boldsymbol{\omega}^{B}\cdot\underline{\boldsymbol{I}}^{B}\cdot{}^{A}\boldsymbol{\omega}^{B}} + \underset{(C40)}{{}^{N}\boldsymbol{\omega}^{A}\cdot\underline{\boldsymbol{I}}^{B}\cdot{}^{A}\boldsymbol{\omega}^{B}} \tag{C41}$$

Equation (13), the formula for the kinetic energy of a cylindrical gyrostat, results directly from substitution of Eq. (C11) into Eq. (C40). After substitution from Eq. (C13) into Eq. (C41), it can be shown that Eq. (C41) applies to spherical gyrostats once J is replaced with I.

Substitution of Eqs. (C11) and (12) into Eq. (C41) leads to

$${}^{N}K^{G} \underset{(C41)(12)(C11)}{=} \tfrac{1}{2}m^{G}\,{}^{N}\boldsymbol{v}^{G_0^2} + \tfrac{1}{2}{}^{N}\boldsymbol{\omega}^{A}\cdot\underline{\boldsymbol{I}}^{G}\cdot{}^{N}\boldsymbol{\omega}^{A} + \tfrac{1}{2}{}^{A}\boldsymbol{\omega}^{B}\cdot J\boldsymbol{bb}\cdot{}^{A}\boldsymbol{\omega}^{B} + {}^{N}\boldsymbol{\omega}^{A}\cdot J\boldsymbol{bb}\cdot{}^{A}\boldsymbol{\omega}^{B} \tag{C42}$$

Replacing ${}^{A}\boldsymbol{\omega}^{B}$ with its right-hand side [see Eq. (C3)] and subsequent rearrangement and substitution of Eq. (C13) gives

$${}^{N}K^{G} \underset{(C42)}{=} \tfrac{1}{2}m^{G}\,{}^{N}\boldsymbol{v}^{G_0^2} + \tfrac{1}{2}{}^{N}\boldsymbol{\omega}^{A}\cdot\underline{\boldsymbol{I}}^{G}\cdot{}^{N}\boldsymbol{\omega}^{A} + {}^{N}\boldsymbol{\omega}^{A}\cdot J\boldsymbol{bb}\cdot\underset{(C3)}{({}^{N}\boldsymbol{\omega}^{B}-{}^{N}\boldsymbol{\omega}^{A})} + \tfrac{1}{2}\underset{(C3)}{({}^{N}\boldsymbol{\omega}^{B}-{}^{N}\boldsymbol{\omega}^{A})}\cdot J\boldsymbol{bb}\cdot\underset{(C3)}{({}^{N}\boldsymbol{\omega}^{B}-{}^{N}\boldsymbol{\omega}^{A})}$$

$$= \tfrac{1}{2}m^{G}\,{}^{N}\boldsymbol{v}^{G_0^2} + \tfrac{1}{2}{}^{N}\boldsymbol{\omega}^{A}\cdot(\underline{\boldsymbol{I}}^{G} - J\boldsymbol{bb})\cdot{}^{N}\boldsymbol{\omega}^{A} + \tfrac{1}{2}({}^{N}\boldsymbol{\omega}^{B}\cdot J\boldsymbol{bb}\cdot{}^{N}\boldsymbol{\omega}^{B})$$

$$\underset{(C12)}{=} \tfrac{1}{2}m^{G}\,{}^{N}\boldsymbol{v}^{G_0^2} + \tfrac{1}{2}{}^{N}\boldsymbol{\omega}^{A}\cdot\underline{\boldsymbol{I}}^{CG}\cdot{}^{N}\boldsymbol{\omega}^{A} + \tfrac{1}{2}({}^{N}\boldsymbol{\omega}^{B}\cdot J\boldsymbol{bb}\cdot{}^{N}\boldsymbol{\omega}^{B}) \tag{C43}$$

$${}^{N}K^{G} \underset{(C14)}{=} {}^{N}K^{CG} + \tfrac{1}{2}J({}^{N}\boldsymbol{\omega}^{B}\cdot\boldsymbol{b})^2 \tag{C44}$$

which results in Eq. (15), the formula for the kinetic energy of a cylindrical gyrostat. Equation (30), the formula for the kinetic energy of a spherical gyrostat, is deduced using a process identical to that shown in Eqs. (C42) and (C43), where $I\underline{\mathbf{1}}$ plays the role of $J\boldsymbol{bb}$.

D. Generalized Momentum

To establish the validity of Eqs. (17), (19), and (43), start by noting that the generalized momentum of any set of ν particles $P_1, \ldots, P_\nu$ moving in a reference frame N is defined (see Ref. 5, page 225) in terms of ${}^{N}\boldsymbol{L}^{P_i}$ (the linear momentum of P_i in N) and ${}^{N}\boldsymbol{v}_r^{P_i}$ (the rth partial velocity of P_i in N) as

$${}^{N}L_r^{S} \triangleq \sum_{i=1}^{\nu} {}^{N}\boldsymbol{v}_r^{P_i}\cdot{}^{N}\boldsymbol{L}^{P_i} \tag{C45}$$

where ${}^{N}\boldsymbol{v}_r^{P_i}$ can be expressed in terms of the partial derivative of ${}^{N}\boldsymbol{v}^{P_i}$ with respect to the motion variable (generalized speed) u_r as

$${}^{N}\boldsymbol{v}_r^{P_i} = \frac{\partial\,{}^{N}\boldsymbol{v}^{P_i}}{\partial u_r} \tag{C46}$$

It has been shown (Ref. 5, page 226) that the generalized momentum is related to kinetic energy by

$${}^{N}L_r^{S} = \frac{\partial\,{}^{N}K^{S}}{\partial u_r} \tag{C47}$$

With Eq. (C47) in hand, the formulas for ${}^{N}L_r^{RG}$ and ${}^{N}L_r^{G}$ are deduced most easily by differentiating the corresponding equations for ${}^{N}K^{RG}$ and ${}^{N}K^{G}$. For example, Eq. (16), the formula for ${}^{N}L_r^{RG}$, is found by substituting Eq. (12) into Eq. (C47) to arrive at

$${}^{N}L_r^{RG} \underset{(C47)}{=} \frac{\partial\,{}^{N}K^{RG}}{\partial u_r} \underset{(12)}{=} \frac{1}{2}\frac{\partial(m^{G}\,{}^{N}\boldsymbol{v}^{G_0}\cdot{}^{N}\boldsymbol{v}^{G_0} + {}^{N}\boldsymbol{\omega}^{A}\cdot\underline{\boldsymbol{I}}^{G}\cdot{}^{N}\boldsymbol{\omega}^{A})}{\partial u_r}$$

$$\underset{(C46)}{=} m^{G}\,{}^{N}\boldsymbol{v}_r^{G_0}\cdot{}^{N}\boldsymbol{v}^{G_0} + {}^{N}\boldsymbol{\omega}_r^{A}\cdot\underline{\boldsymbol{I}}^{G}\cdot{}^{N}\boldsymbol{\omega}^{A}$$

$$= {}^{N}\boldsymbol{v}_r^{G_0}\cdot\underset{(C55)}{{}^{N}\boldsymbol{L}^{G}} + {}^{N}\boldsymbol{\omega}_r^{A}\cdot\underset{(3)}{{}^{N}\boldsymbol{H}^{RG/G_0}} \tag{C48}$$

Similarly, Eq. (17), the formula for the generalized momentum of cylindrical gyrostats, can be found by substituting Eq. (13) into Eq. (C47) and subsequently making use of Eq. (C48) to produce

$${}^{N}L_r^{G} \underset{(C47)}{=} \frac{\partial\,{}^{N}K^{G}}{\partial u_r} \underset{(13)}{=} \frac{\partial\left({}^{N}K^{RG} + \frac{1}{2}J\,{}^{A}\boldsymbol{\omega}^{B}\cdot{}^{A}\boldsymbol{\omega}^{B} + J\,{}^{N}\boldsymbol{\omega}^{A}\cdot{}^{A}\boldsymbol{\omega}^{B}\right)}{\partial u_r}$$

$$= \underset{(96)}{{}^{N}L^{RG}} + J\left({}^{A}\boldsymbol{\omega}_r^{B}\cdot{}^{A}\boldsymbol{\omega}^{B} + {}^{A}\boldsymbol{\omega}_r^{B}\cdot{}^{N}\boldsymbol{\omega}^{A} + {}^{N}\boldsymbol{\omega}_r^{A}\cdot{}^{A}\boldsymbol{\omega}^{B}\right)$$

$$\underset{(C1)}{=} {}^{N}L^{RG} + J\left({}^{A}\boldsymbol{\omega}_r^{B}\cdot{}^{N}\boldsymbol{\omega}^{B} + {}^{N}\boldsymbol{\omega}_r^{A}\cdot{}^{A}\boldsymbol{\omega}^{B}\right) \tag{C49}$$

E. Linear Momentum and Effective Force

Although formulas for linear momentum and effective force were not presented in the result section, they provide a simple conduit for the derivation of the generalized effective force of a gyrostat. The linear momentum ${}^{N}\boldsymbol{L}^{S}$ of a system S of ν particles $A_1, \ldots, A_\nu$ moving in a reference frame N is defined in terms of m^{P_i} (the mass of P_i and ${}^{N}\boldsymbol{v}^{P_i}$) as

$${}^{N}\boldsymbol{L}^{S} \triangleq \sum_{i=1}^{\nu} m^{P_i}\,{}^{N}\boldsymbol{v}^{P_i} \tag{C50}$$

Here ${}^{N}\boldsymbol{v}^{P_i}$ may be expressed in terms of ${}^{N}\boldsymbol{v}^{S_0}$ (the velocity of the mass center of S) and the time derivative of $\boldsymbol{p}^{S_0P_i}$ (the position vector of P_i from S_0) as

$${}^{N}\boldsymbol{v}^{P_i} = {}^{N}\boldsymbol{v}^{S_0} + \frac{{}^{N}\mathrm{d}\boldsymbol{p}^{S_0P_i}}{\mathrm{d}t} \tag{C51}$$

Substitution of Eq. (C51) into Eq. (C50) yields

$${}^{N}\boldsymbol{L}^{S} \underset{(C50)(C51)}{=} {}^{N}\boldsymbol{v}^{S_0}\sum_{i=1}^{\nu} m^{P_i} + \frac{{}^{N}\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\nu} m^{P_i}\boldsymbol{p}^{S_0P_i} \tag{C52}$$

The first term in Eq. (C50) can be simplified by noting that m^{S} is by definition

$$\sum_{i=1}^{\nu} m^{P_i}$$

Setting the second term in Eq. (C50) equal to $\mathbf{0}$ is in fact reiterating the definition that states that S_0 is the mass center of S. Combining these two facts produces the familiar formula for the linear momentum of an arbitrary set of particles, namely,

$${}^{N}\boldsymbol{L}^{S} \underset{(C52)}{=} m^{S}\,{}^{N}\boldsymbol{v}^{S_0} \tag{C53}$$

The effective force ${}^{N}\boldsymbol{F}^{S}$ of a system S of ν particles $P_1, \ldots, A_\nu$ moving in a reference frame N is defined in terms of m^{P_i} (the mass of P_i) and ${}^{N}\boldsymbol{a}^{P_i}$ as

$${}^{N}\boldsymbol{F}^{S} \triangleq \sum_{i=1}^{\nu} {}^{N}\boldsymbol{F}^{P_i} = \sum_{i=1}^{\nu} m^{P_i}\,{}^{N}\boldsymbol{a}^{P_i} \tag{C54}$$

Comparison of Eqs. (C50) and (C54) leads to the conclusion

$$^N\boldsymbol{F}^S \underset{(C50)(C54)}{=} \frac{^N\mathrm{d}\,^N\boldsymbol{L}^S}{\mathrm{d}t} \tag{C55}$$

Combining Eqs. (C53) and (C55) produces the familiar formula for the effective force of an arbitrary set of particles, namely,

$$^N\boldsymbol{F}^S \underset{(C53)(C55)}{=} m^S\,{}^N\boldsymbol{a}^{S_0} \tag{C56}$$

F. Generalized Effective Forces

To establish the validity of Eqs. (21), (23), (45), and (47), one starts by noting that the generalized effective force in a reference frame N of a set of α particles $A_1, \ldots, A_\alpha$ of a rigid body A and a set of β particles $B_1, \ldots, B_\beta$ of a rigid body B are defined in terms of $^N\boldsymbol{v}_r^{A_i}$ (the rth partial velocity of A_i in N), $^N\boldsymbol{F}^{A_i}$ (the effective force of A_i in N), $^N\boldsymbol{v}_r^{B_i}$ (the rth partial velocity of B_i in N), and $^N\boldsymbol{F}^{B_i}$ (the effective force of B_i in N) as

$$^NF_T^G \triangleq \sum_{i=1}^{\alpha} {}^N\boldsymbol{v}_r^{A_i} \cdot {}^N\boldsymbol{F}^{A_i} + \sum_{i=1}^{\beta} {}^N\boldsymbol{v}_r^{B_i} \cdot {}^N\boldsymbol{F}^{B_i} \tag{C57}$$

A useful relationship for $^N\boldsymbol{v}_r^{A_i}$, following from the formula [see Ref. 5, page 30, Eq. (2.7.1)] relating velocities of two points fixed on the same rigid body, is

$$^N\boldsymbol{v}_r^{A_i} = {}^N\boldsymbol{v}_r^{G_0} + {}^N\boldsymbol{\omega}_r^A \times \boldsymbol{p}^{G_0A_i} \tag{C58}$$

A related useful relationship for $^N\boldsymbol{v}_r^{B_i}$ that follows from the formula relating the velocity of a point moving on A is

$$^N\boldsymbol{v}_r^{B_i} = {}^N\boldsymbol{v}_r^{G_0} + {}^N\boldsymbol{\omega}_r^A \times \boldsymbol{p}^{G_0B_i} + {}^A\boldsymbol{v}_r^{B_i} \tag{C59}$$

Substitution of Eqs. (C58) and (C59) into Eq. (C57) and subsequent rearrangement produces

$$\begin{aligned} ^NF_r^G \underset{(C57–C59)}{=} {}& {}^N\boldsymbol{v}_r^{G_0} \cdot \left(\sum_{i=1}^{\alpha} {}^N\boldsymbol{F}^{A_i} + \sum_{i=1}^{\beta} {}^N\boldsymbol{F}^{B_i} \right) \\ & + {}^N\boldsymbol{\omega}_r^A \cdot \left(\sum_{i=1}^{\alpha} \boldsymbol{p}^{G_0A_i} \times {}^N\boldsymbol{F}^{A_i} + \sum_{i=1}^{\beta} \boldsymbol{p}^{G_0B_i} \times {}^N\boldsymbol{F}^{B_i} \right) \\ & + \sum_{i=1}^{\beta} {}^A\boldsymbol{v}_r^{B_i} \cdot {}^N\boldsymbol{F}^{B_i} \end{aligned} \tag{C60}$$

The summations in the first term in Eq. (C60) are by definition [see Eq. (C54)] the effective force of G in N. In light of Eq. (C56), this summation can be rewritten as

$$^N\boldsymbol{F}^G \triangleq \sum_{i=1}^{\alpha} {}^N\boldsymbol{F}^{A_i} + \sum_{i=1}^{\beta} {}^N\boldsymbol{F}^{B_i} \underset{(C56)}{=} m^G\,{}^N\boldsymbol{a}^{G_0} \tag{C61}$$

The summations in the second term in Eq. (C60) are by definition [see Eq. (C28)] the effective torque of G in N:

$$^N\boldsymbol{T}^G \triangleq \sum_{i=1}^{\alpha} \boldsymbol{p}^{G_0A_i} \times {}^N\boldsymbol{F}^{A_i} + \sum_{i=1}^{\beta} \boldsymbol{p}^{G_0B_i} \times {}^N\boldsymbol{F}^{B_i} \tag{C62}$$

Substitution of Eqs. (C61) and (C62) into Eq. (C60) and subsequent substitution of Eq. (C5) gives

$$\begin{aligned} ^NF_r^G & \underset{(C60)}{=} {}^N\boldsymbol{v}_r^{G_0} \cdot \underset{(C61)}{^N\boldsymbol{F}^G} + {}^N\boldsymbol{\omega}_r^A \cdot \underset{(C62)}{^N\boldsymbol{T}^G} + \sum_{i=1}^{\beta} m^{B_i}\,{}^A\boldsymbol{v}_r^{B_i} \cdot {}^N\boldsymbol{a}^{B_i} \\ & \underset{(C5)}{=} {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{T}^G + {}^A\boldsymbol{\omega}_r^B \cdot \sum_{i=1}^{\beta} m^{B_i}\boldsymbol{r}_i \times {}^N\boldsymbol{a}^{B_i} \end{aligned} \tag{C63}$$

It can be shown (Ref. 5, pages 126 and 127) that

$$\sum_{i=1}^{\beta} m^{B_i}\boldsymbol{r}_i \times {}^N\boldsymbol{a}^{B_i}$$

which appears in the last term in Eq. (C63), may be cast in terms of $\underline{\boldsymbol{I}}^B$ (the central inertia dyadic of B) and the angular velocity and acceleration of A in N as

$$\sum_{i=1}^{\beta} m^{B_i}\boldsymbol{r}_i \times {}^N\boldsymbol{a}^{B_i} = \underline{\boldsymbol{I}}^B \cdot {}^N\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^B \times \underline{\boldsymbol{I}}^B \cdot {}^N\boldsymbol{\omega}^B \tag{C64}$$

Hence, Eq. (C63) can be reformed as

$$\begin{aligned} ^NF_r^G \underset{(C63)}{=} {}& {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{T}^G \\ & + {}^A\boldsymbol{\omega}_r^B \cdot (\underline{\boldsymbol{I}}^B \cdot {}^N\boldsymbol{\alpha}^B + \underset{(C64)}{^N\boldsymbol{\omega}^B} \times \underline{\boldsymbol{I}}^B \cdot {}^N\boldsymbol{\omega}^B) \end{aligned} \tag{C65}$$

1. *Proof of Generalized Effective Force for Cylindrical Gyrostats*

The third term in Eq. (C65) can be further simplified for cylindrical gyrostats. Substitution of Eq. (C10) into the last term in the parentheses in Eq. (C63) gives

$$\begin{aligned} & {}^A\boldsymbol{\omega}_r^B \cdot ({}^N\boldsymbol{\omega}^B \times \underline{\boldsymbol{I}}^B \cdot {}^N\boldsymbol{\omega}^B) \\ & \quad \underset{(C10)}{=} (J-K)\,{}^A\boldsymbol{\omega}_r^B \cdot ({}^N\boldsymbol{\omega}^B \times \boldsymbol{b})(\boldsymbol{b} \cdot {}^N\boldsymbol{\omega}^B) \underset{(C9)}{=} 0 \end{aligned} \tag{C66}$$

The first term in parentheses in Eq. (C65) can be further simplified using Eq. (C11)

$${}^A\boldsymbol{\omega}_r^B \cdot \underline{\boldsymbol{I}}^B \cdot {}^N\boldsymbol{\alpha}^B \underset{(C11)}{=} J\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B \tag{C67}$$

Combining the information in Eqs. (C66) and (C67) into Eq. (C65) leads to

$$^NF_r^G \underset{(C65)}{=} {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{T}^G + \underset{(C66)(C67)}{J\,{}^A\boldsymbol{\omega}_r^B \cdot \underline{\boldsymbol{I}}^B \cdot {}^N\boldsymbol{\alpha}^B} \tag{C68}$$

Expressing $^N\boldsymbol{T}^G$ in terms of $^N\boldsymbol{T}^{RG}$ leads to

$$\begin{aligned} ^NF_r^G & \underset{(C68)}{=} {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot [{}^N\boldsymbol{T}^{RG} + J({}^A\boldsymbol{\alpha}^B + \underset{(8)}{^N\boldsymbol{\omega}^A} \times {}^A\boldsymbol{\omega}^B)] \\ & \qquad + J\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B \\ & = {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{T}^{RG} \\ & \qquad + J\,{}^N\boldsymbol{\omega}_r^A \cdot ({}^A\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B) + J\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B \\ & = \underset{(20)}{^NF_r^{RG}} + J\,{}^N\boldsymbol{\omega}_r^A \cdot ({}^A\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B) + J\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B \end{aligned} \tag{C69}$$

As will become useful momentarily, differentiation of Eq. (C1) in N, leads to the angular acceleration addition theorem, namely,

$$^N\boldsymbol{\alpha}^B \underset{(C1)}{=} {}^N\boldsymbol{\alpha}^A + {}^A\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B \tag{C70}$$

In view of Eq. (C70), the third term in Eq. (C69) can be reexpressed as

$$\begin{aligned} J\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B & \underset{(C70)}{=} J\,{}^A\boldsymbol{\omega}_r^B \cdot ({}^N\boldsymbol{\alpha}^A + {}^A\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B) \\ & \underset{(C9)}{=} J\,{}^A\boldsymbol{\omega}_r^B \cdot ({}^N\boldsymbol{\alpha}^A + {}^A\boldsymbol{\alpha}^B) \end{aligned} \tag{C71}$$

Making use of Eq. (C71) in Eq. (C69) results in

$$\begin{aligned} ^NF_r^G \underset{(C69)}{=} {}& {}^NF_r^{RG} + J\,{}^N\boldsymbol{\omega}_r^A \cdot ({}^A\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B) \\ & + \underset{(C71)}{J\,{}^A\boldsymbol{\omega}_r^B \cdot ({}^N\boldsymbol{\alpha}^A + {}^A\boldsymbol{\alpha}^B)} \end{aligned} \tag{C72}$$

which is Eq. (21), one of the formulas for the generalized effective force of a cylindrical gyrostat. The alternate expression for the generalized effective force of a cylindrical gyrostats is derived from Eq. (C68) by expressing ${}^N\boldsymbol{T}^G$ in terms of ${}^N\boldsymbol{T}^{CG}$ as

$$
\begin{aligned}
{}^NF_r^G &\underset{(C68)}{=} {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot [{}^N\boldsymbol{T}^{CG} \\
&\quad + J(\boldsymbol{bb} \cdot \underset{(10)}{{}^N\boldsymbol{\alpha}^B} + {}^N\boldsymbol{\omega}^A \times \boldsymbol{bb} \cdot {}^N\boldsymbol{\omega}^B)] + J\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B \\
&= {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{T}^{CG} + J\,{}^N\boldsymbol{\omega}_r^A \cdot (\boldsymbol{bb} \cdot {}^N\boldsymbol{\alpha}^B \\
&\quad + {}^N\boldsymbol{\omega}^A \times \boldsymbol{bb} \cdot {}^N\boldsymbol{\omega}^B) + J\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B \\
&= \underset{(22)}{{}^NF_r^{CG}} + J\,{}^N\boldsymbol{\omega}_r^A \cdot (\boldsymbol{bb} \cdot {}^N\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times \boldsymbol{bb} \cdot {}^N\boldsymbol{\omega}^B) \\
&\quad + J\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B \\
&= {}^NF_r^{CG} + J\,{}^N\boldsymbol{\omega}_r^B \cdot \boldsymbol{bb} \cdot {}^N\boldsymbol{\alpha}^B + J\,{}^N\boldsymbol{\omega}_r^A \cdot ({}^N\boldsymbol{\omega}^A \times \boldsymbol{bb} \cdot {}^N\boldsymbol{\omega}^B)
\end{aligned}
\tag{C73}
$$

which results in Eq. (23).

2. Proof of Generalized Effective Force for Spherical Gyrostats

Substitution of Eq. (C13) into Eq. (C65) and subsequent use of Eq. (C14) gives

$$
\begin{aligned}
{}^NF_r^G &\underset{(C65)}{=} {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{T}^G \\
&\quad + {}^A\boldsymbol{\omega}_r^B \cdot (\underset{(C13)}{I\boldsymbol{1}} \cdot {}^N\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^B \times \underset{(C13)}{I\boldsymbol{1}} \cdot {}^N\boldsymbol{\omega}^B) \\
&\underset{(C14)}{=} {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{T}^G + I\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B
\end{aligned}
\tag{C74}
$$

Expressing ${}^N\boldsymbol{T}^G$ in terms of ${}^N\boldsymbol{T}^{RG}$ leads to

$$
\begin{aligned}
{}^NF_r^G &\underset{(C74)}{=} {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot [{}^N\boldsymbol{T}^{RG} + I({}^A\boldsymbol{\alpha}^B \\
&\quad + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B)] + I\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B \\
&= {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{T}^{RG} + I\,{}^N\boldsymbol{\omega}_r^A \cdot ({}^A\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B) \\
&\quad + I\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B \\
&= \underset{(44)}{{}^NF_r^{RG}} + I\,{}^N\boldsymbol{\omega}_r^A \cdot ({}^A\boldsymbol{\alpha}^B + {}^N\boldsymbol{\omega}^A \times {}^A\boldsymbol{\omega}^B) + I\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B
\end{aligned}
\tag{C75}
$$

which is Eq. (45), one of the formulas for the generalized effective force of a spherical gyrostat. The alternate expression for the generalized effective force of a spherical gyrostats is derived from Eq. (C74) by expressing ${}^N\boldsymbol{T}^G$ in terms of ${}^N\boldsymbol{T}^{SG}$ as

$$
\begin{aligned}
{}^NF_r^G &\underset{(C74)}{=} {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot ({}^N\boldsymbol{T}^{SG} \underset{(34)}{+} I\,{}^N\boldsymbol{\alpha}^B) + I\,{}^A\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B \\
&= {}^N\boldsymbol{v}_r^{G_0} \cdot {}^N\boldsymbol{F}^G + {}^N\boldsymbol{\omega}_r^A \cdot {}^N\boldsymbol{T}^{SG} + I\,{}^N\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B \\
&= \underset{(46)}{{}^NF_r^{SG}} + I\,{}^N\boldsymbol{\omega}_r^B \cdot {}^N\boldsymbol{\alpha}^B
\end{aligned}
\tag{C76}
$$

which results in Eq. (47).

References

[1] Sayers, M. W., "Symbolic Computer Methods to Automatically Formulate Vehicle Simulation Codes," Ph.D. Thesis, Dept. of Mechanical Engineering, Univ. of Michigan, Ann Arbor, MI, Feb. 1990.

[2] Schaechter, D. B., and Levinson, D. A., "Interactive Computerized Symbolic Dynamics for the Dynamicist," *Journal of the Astronautical Sciences*, Vol. 36, No. 4, 1988, pp. 365–388.

[3] Mitiguy, P. C., and Kane, T. R., "Motion Variables Leading to Efficient Equations of Motion," *International Journal of Robotics Research*, Vol. 15, No. 5, 1996, pp. 522–532.

[4] Roberson, R. E., "The Equivalence of Two Classical Problems of Free Spinning Gyrostats," *Journal of Applied Mechanics*, Vol. 38, No. 3, 1971, pp. 707, 708.

[5] Kane, T. R., and Levinson, D. A., *Dynamics: Theory and Applications*, McGraw–Hill, New York, 1985, pp. 24, 30, 32, 69, 70, 125–127, 147, 148, 225, and 226.

[6] Fox, L., *Numerical Solutions of Ordinary and Partial Differential Equations*, Addison Wesley, Palo Alto, CA, 1962, pp. 24, 25.

[7] Hughes, P. C., *Spacecraft Attitude Dynamics*, Wiley, New York, 1986, pp. 67, 158, and 178.

[8] Kane, T. R., Likins, P. W., and Levinson, D. A., *Spacecraft Dynamics*, McGraw–Hill, New York, 1983, p. 212.

[9] Wittenburg, J., *Dynamics of Systems of Rigid Bodies*, B. G. Teubner, Stuttgart, Germany, 1977, p. 67.

[10] Kane, T. R., "Angular Momentum, Kinetic Energy, and Generalized Inertia Forces for a Gyrostat," *Journal of Applied Mechanics*, Vol. 43, No. 3, 1976, pp. 515–517.

[11] Arnold, R. N., and Maunder, L., *Gyrodynamics and its Engineering Applications*, Academic, New York, 1961.

[12] Gray, A., *A Treatise on Gyrostatics and Rotational Motion*, Dover, New York, 1959.

[13] Greenhill, G., *Gyroscopic Theory*, Chelsea, New York, 1966.

[14] Leimanis, E., *The General Problem of the Motion of Coupled Rigid Bodies about a Fixed Point*, Springer-Verlag, New York, 1965.

[15] Scarborough, J. B., *The Gyroscope: Theory and Applications*, Interscience, New York, 1958.